CERTAIN PROBLEMS OF FINSLER SPACES

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By

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This is to certify that **Sunita Pal** worked under my supervision for the D. Phil. degree on the problem entitled **"CERTAIN PROBLEMS OF FINSLER SPACES"** since August 29, 2000. The present thesis submitted by her embodies the work done by the candidate herself. The work in hand has not been submitted for the award of any other degree.

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PREFACE

The present thesis entitled "CERTAIN PROBLEMS OF FINSLER SPACES" is an outcome of my researches done during the last two years at the University of Allahabad.

The thesis is divided into five chapters and each chapter is divided into several sections. The decimal notation has been employed for numbering the equations. References to the equations are of the form (C.S.E.), where C, S and E stand for the corresponding chapter, section and equation respectively. If C coincides with the chapter at hand, it is omitted. The numbers in the square bracket refer to the references given at the end. The notation for the skew-symmetric part is given by $-k/h$ which means the subtraction of the terms obtained by interchanging the indices k and h from the former terms, e.g. Ω_{kh} $-k/h$ $= \Omega_{kh} - \Omega_{hk}$. The partial differential operators with respect to x^k and y^k have been denoted by ∂_k and $\dot{\partial}_k$ respectively. Cartan v-covariant derivative with respect to y^k, Cartan h-covariant derivative with respect to x^k, Berwald covariant derivative with respect to x^k, Lie derivative with respect to an infinitesimal transformation and δ-covariant derivative with respect to u^σ are denoted by $|_k$, $_{|k}$, $\mathcal{B}_k$, £ and $;_\sigma$ respectively. The symbol $p.$ stands for the projection on indicatrix.

The first chapter of the thesis is introductory and it includes the concepts, definition and formulae which are used in subsequent chapters.

The second chapter deals with a Finsler space whose curvature tensor R^i_{jkh} is generalized recurrent i.e., satisfies the condition $\alpha R^i_{jkh|m|l} + \beta_l R^i_{jkh|m} + \gamma_m R^i_{jkh|l} + v_{lm} R^i_{jkh}$ $= 0$. Such space has been named as R^h-GR space. We prove that R^h-recurrent Finsler spaces, R^h-birecurrent Finsler spaces, R^h-generalized birecurrent Finsler spaces of first and second kind and R^h-special generalized birecurrent spaces of first and second kind are particular cases of an R^h-GR space. In different sections of this chapter various theorems concerned with such space have been established. Certain identities in a P2-like

R^h-GR space have been derived and various results concerning projection on indicatrix have been obtained.

The third chapter is devoted to the study of special projective motions. This chapter is divided into seven sections. The first and second sections are of introductory nature and deal with the concept of projective motion. Rest five sections deal with projective motions generated by vector fields satisfying some generalized conditions. Certain theorems concerning these types of projective motions have been established.

Fourth chapter is on hypersurface of special Finsler spaces. The first two sections of this chapter present a historical background and development of theory of the hypersurface. In the next two sections, we define C^δ-recurrent and C^δ-birecurrent Finsler spaces and study some properties of these spaces. We also discuss the hypersurface of a C^h-recurrent and C^h-birecurrent Finsler spaces. Some results concerning totally geodesic and umbilical hypersurface of such spaces have been obtained. We study the hypersurface of a C2-like Finsler space in the last section.

The fifth chapter is devoted to the study of a hypersurface of a recurrent Finsler space equipped with Berwald connection. Several results have been obtained for hypersurface of such space.

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Chapter I

FINSLER SPACES

1. Introduction

Let R be a region of an n-dimensional space X_n, which is completely covered with a coordinate system such that each point P of R is represented by n-tuples x^i $(i = 1,2,\cdots n)$ of real numbers called *coordinates* of P. Consider a curve $C : x^i = x^i(t)$ passing through x. The entities $y^i = \dfrac{dx^i}{dt}$ constitute the components of the tangent vector to the curve C at the point $P(x^i)$. The combination (x^i, y^i) which is conveniently written as (x, y) is known as *line-element* of the curve C with centre at P [120]. x^i and y^i are called *positional* and *directional coordinates* respectively.

Let $P(x^i)$ and $Q(x^i + dx^i)$ be two neighbouring points of the region R. The infinitesimal distance ds between the points P and Q is defined by

$$ds := F(x, dx), \tag{1.1}$$

where $F(x, dx)$ is a function defined for all line elements (x, y) in the region R and satisfies the following conditions [120]

Condition (a): The function $F(x, y)$ is positively homogeneous of degree one in y^i, i.e.

$$F(x, ky) = kF(x, y), \tag{1.2}$$

where k is some positive scalar.

Condition (b): The function $F(x, y)$ is positive unless all y^i vanish simultaneously, i.e.

(1.3) $$F(x,y) > 0, \quad \text{with} \sum_i (y^i)^2 \neq 0 \; .$$

Condition (c): The quadratic form

(1.4) $$\{\dot\partial_i \dot\partial_j F^2(x,y)\} X^i X^j, \quad \dot\partial_i \equiv \partial / \partial y^i ,$$

is assumed to be positive definite for all variables X^i.

Definition (1.1). An *n*-dimensional space X_n equipped with such a function $F(x,y)$ satisfying the above three conditions is called a Finsler space, and we shall denote it by F_n. The function $F(x,y)$ is called as the fundamental function of the Finsler space F_n.

2. Metric Tensor

The quantities $g_{ij}(x,y)$, defined by

(2.1) $$g_{ij}(x,y) = \frac{1}{2} \dot\partial_i \dot\partial_j F^2(x,y) ,$$

form the components of a covariant tensor of rank 2, called m*etric tensor.* From (2.1) it is obvious that the metric tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i and symmetric in *i* and *j*. In view of Euler's theorem on homogeneous function, we have

(2.2) $$g_{ij}(x,y) y^i y^j = F^2(x,y) .$$

Again due to Euler's theorem on homogeneous function, the derivative of *F* satisfy the following:

(2.3) a) $$y^i \dot\partial_i F(x,y) = F(x,y) ,$$

b) $$y^i \dot\partial_i \dot\partial_j F(x,y) = 0 .$$

Therefore from (2.2) we may express the infinitesimal distance ds between two neighbouring points x and $x+dx$ in terms of the metric tensor $g_{ij}(x, y)$ as follows

$$ds^2 = g_{ij}(x, dx)dx^i dx^j . \tag{2.4}$$

3. The Tangent Space and Its Dual Space

Let us consider a change of local coordinates represented by

$$\bar{x}^i = \bar{x}^i(x^j(t)) . \tag{3.1}$$

The components $y^i = \dfrac{dx^i}{dt}$ of the tangent vector are transformed according to

$$\bar{y}^i = (\partial_j \bar{x}^i) y^j , \qquad \partial_i \equiv \frac{\partial}{\partial x^i} , \tag{3.2}$$

which in terms of differentials are written as

$$d\bar{x}^i = (\partial_j \bar{x}^i) dx^j . \tag{3.3}$$

A system of n-quantities X^i is called a *contravariant vector* attached to the point $P(x^i)$ of F_n if its transformation law under (3.1) is similar to that of y^i, the individual X^i represent its components. Such contravariant vectors attached to $P(x^i)$ constitute the elements of a vector space called *tangent space* at $P(x^i)$ and is denoted by $T_n(P)$ or $T_n(x^i)$. The length of an arbitrary vector η^i of $T_n(P)$ is given by $F(x^i, \eta^i)$. In view of (2.2), all lengths in $T_n(P)$ may be expressed in terms of the g_{ij} defined by (2.1), which we shall regard as the components of the metric tensor of $T_n(P)$.

To each contravariant vector y^i of the tangent space T_n(P), there corresponds a covariant vector y_i such that

(3.4) $$y_i = g_{ij} y^j \ .$$

The set of all such covariant vectors associated with the point P of F_n forms a vector space called as the *dual tangent space* at P and is denoted by $T_n'(P)$.

The *Hamiltonian function* $H(x^i, y_i)$ satisfying the three conditions required for the fundamental function of a Finsler space constitute the metric function of the dual space $T_n'(P)$.

Analogous to the metric tensor $g_{ij}(x, y)$, we define a tensor $g^{ij}(x^k, y_k)$ as follows:

(3.5) $$g^{ij}(x^k, y_k) := \frac{1}{2} \bar{\partial}_i \bar{\partial}_j H^2(x^k, y_k)$$

where $\bar{\partial}_i$ denotes the partial differentiation with respect to the covariant vector y_i. The quantities $g^{ij}(x^k, y_k)$ constitute the components of a contravariant tensor of rank 2.

4. Properties of the Metric Tensor

The quantities g_{ij} and g^{ij}, defined by (2.1) and (3.5), are connected by

(4.1) $$g_{ij} g^{jk} = \delta_i^k \ \begin{cases} 1 \text{ if } k = i \\ 0 \text{ if } k \neq i \end{cases}$$

Using Euler's theorem on homogeneous function, we may derive the following from (2.1):

(4.2) $$g_{ij} y^j = \frac{1}{2} \dot{\partial}_i F^2 = F \dot{\partial}_i F \ .$$

Using equation (3.4) in (4.2), we get

(4.3) $$y_i = F \dot{\partial}_i F \ .$$

The vector y_i also satisfies

(4.4) a) $$y_i y^i = F^2,$$

b) $$g_{ij} = \dot{\partial}_i y_j .$$

5. The $(h)hv$-Torsion Tensor and Generalized Christoffel Symbols

From the metric tensor we construct a new tensor C_{ijk} by differentiating (2.1) partially with respect to y^k. This new tensor C_{ijk}, defined by

(5.1) $$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij} = \frac{1}{4}\dot{\partial}_k \dot{\partial}_i \dot{\partial}_j F^2,$$

is known as *$(h)hv$-torsion tensor* [39]. It is positively homogeneous of degree -1 in y^i and symmetric in all its indices.

By Euler's theorem on homogeneous function, we get the following identities

(5.2) a) $$C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0,$$

b) $$C^i_{jk} y_i = C^i_{kj} y_i = 0$$

and

c) $$C^i_{jk} y^j = C^i_{kj} y^j = 0,$$

where C^i_{jk}, the associate tensor of C_{ijk}, is defined by

(5.3) $$C^h_{ik} := g^{hj} C_{ijk}.$$

This tensor is also positively homogeneous of degree -1 in y^i and symmetric in its lower indices.

As in case of Riemannian geometry, here also we define *generalized Christoffel Symbols* of the *first* and *second kind* as follows:

(5.4) a) $$\gamma_{ijk} := \frac{1}{2}(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik})$$

and

b) $$\gamma^h_{ik} := g^{hj}\gamma_{ijk}.$$

6. Euclidean Connection of Cartan

Due to the first postulate of E. Cartan the square of the magnitude of an arbitrary vector field X^i is defined as $g_{ij}X^iX^j$. In view of this fact the equation (2.2) shows that the function F is the magnitude of the vector y^i. Hence the unit vector l^i in the direction of y^i is given by

(6.1) a) $$l^i := \frac{y^i}{F}$$

and the associate vector of l^i is defined by

(6.1) b) $$l_i := g_{ij}l^j = \partial_i F = \frac{y_i}{F}.$$

This associate vector is called the *normalized supporting element.*

The vector l_i obviously satisfies

(6.2) $$l^i l_i = 1.$$

Cartan, in his second postulate represented the variation of an arbitrary vector field X^i under an infinitesimal change of its line-element (x, y) to $(x+dx, y+dy)$ by means of a covariant (or absolute) differential [14, 15, 120]

(6.3) a) $$DX^i = dX^i + X^j(C^i_{jk}dy^k + \Gamma^i_{jk}dx^k),$$

where

(6.3) b) $$\Gamma^i_{jk} := \gamma^i_{jk} - C^i_{mk}G^m_j + g^{ih}C_{jkm}G^m_h$$

together with

(6.3) c) $$G^i := \frac{1}{2}\gamma^i_{jk}y^j y^k$$

and

(6.3) d) $$G^i_h := \dot{\partial}_h G^i.$$

The function G^i are positively homogeneous of degree two in y^i. Eliminating dy^k from equation (6.3a) and the absolute differential of l^i, E. Cartan deduced [14, 15]

(6.4) $$DX^i = FX^i|_k\, Dl^k + X^i_{|k}dx^k + y^k(\dot{\partial}_k X^i)\frac{dF}{F},$$

where

(6.5) $$X^i|_k := \dot{\partial}_k X^i + X^r C^i_{rk},$$

(6.6) $$X^i_{|k} := \partial_k X^i - (\dot{\partial}_r X^i)G^r_k + X^r\Gamma^{*i}_{rk},$$

(6.7) $$\Gamma^{*i}_{rk} := \Gamma^{i}_{rk} - C^{i}_{mr}\Gamma^{m}_{sk}y^{s}$$

and

(6.8) $$\Gamma^{*r}_{sk}y^{s} = G^{r}_{k}.$$

The function Γ^{*i}_{rk}, defined by (6.7), are *connection parameters* of E. Cartan. These are symmetric in lower indices and are positively homogeneous of degree zero in y^{i}. The equations (6.5) and (6.6) give two processes of covariant differentiation called *v-covariant differentiation* and *h-covariant differentiation* respectively. $X^{i}\,|_{h}$ and $X^{i}_{|h}$ are respectively *v*-covariant derivative and *h*-covariant derivative of the vector field X^{i}. We must note that the expression for $X^{i}\,|_{h}$ taken by E. Cartan differs from expression written here, though the notations are the same. This expression for $X^{i}\,|_{h}$ is due to Makoto Matsumoto. In fact, $X^{i}\,|_{h}$ of E. Cartan is equal to $FX^{i}\,|_{h}$ of Makoto Matsumoto [36-38, 40, 41, 43-51]. K. Yano [147, 148] denoted the above covariant derivatives by $\dot{\nabla}_{h}X^{i}$ and $\nabla_{h}X^{i}$ respectively.

In particular, the metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constants with respect to above processes, i.e.

(6.9) a) $g_{ij}\,|_{k} = 0,$ b) $g^{ij}\,|_{k} = 0,$

and

(6.10) a) $g_{ij|k} = 0,$ b) $g^{ij}_{|k} = 0.$

The vector y^{i}, l^{i} and the metric function F vanish under *h*-covariant differentiation, i.e.

(6.11) a) $$y^{i}_{|j} = 0,$$

b) $$l^i_{|j} = 0$$

and

c) $$F_{|k} = 0\,.$$

The two processes of covariant differentiation defined above commute with the process of partial differentiation with respect to y^j according as

(6.12) a) $$\dot\partial_j(X^i|_k) - (\dot\partial_j X^i)|_k = X^s(\dot\partial_j C^i_{ks}) + C^s_{kj}(\dot\partial_s X^i)$$

and

b) $$\dot\partial_j(X^i_{|k}) - (\dot\partial_j X^i)_{|k} = X^s(\dot\partial_j \Gamma^{*i}_{sk}) - (\dot\partial_s X^i)P^s_{jk}\,,$$

where

c) $$P^s_{jk} := (\dot\partial_j \Gamma^{*s}_{hk})y^h = \Gamma^{*s}_{jhk}y^h\,.$$

7. Berwald's Covariant Differentiation

L. Berwald considered new connection parameters G^i_{jk} which are connected with Cartan's connection parameters Γ^{*i}_{jk} by the equation

$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|s}y^s\,. \tag{7.1}$$

Berwald's connection coefficients G^i_{jk} are positively homogeneous of degree zero in y^i and satisfy

$$G^i_{jk} = \dot\partial_j\,\dot\partial_k\,G^i\,, \tag{7.2}$$

where

$$G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k .$$

Similar to the Cartan's covariant derivatives L. Berwald defined covariant derivative for his connection parameters G^i_{jk}. The Berwald's covariant derivative of an arbitrary tensor field T^i_j with respect to x^k is given by

(7.3) $$\mathcal{B}_k T^i_j := \partial_k T^i_j - (\dot{\partial}_s T^i_j) G^s_k + T^s_j G^i_{sk} - T^i_s G^s_{jk} .$$

In most of the existing literature Berwald covariant derivative $\mathcal{B}_k T^i_j$ appears as $T^i_{j(k)}$. However the notation $\mathcal{B}_k T^i_j$, is quite familiar [24, 55-61, 80, 82, 83, 85-91, 93-102, 105]. Transvecting (7.2) by y^j and using Euler's theorem on homogeneous function, we get

(7.4) a) $$G^i_{jk} y^j = G^i_k = \dot{\partial}_k G^i$$

and

b) $$G^i_k y^k = G^i_{jk} y^j y^k = 2G^i .$$

In view of homogeneity of G^i_{jk} in y^i and equation (7.4) the functions G^i and G^i_k are positively homogeneous of degree two and one in y^i respectively. Berwald's connection coefficients G^i_{jk} do not form the components of a tensor, but their partial derivatives with respect to y^h constitute the components of a tensor. Thus

(7.5) $$G^i_{hjk} := \dot{\partial}_h G^i_{jk}$$

form a tensor which is symmetric in all its lower indices and positively homogeneous of degree -1 in y^i. Thus

(7.6) $$G^i_{jkh} = G^i_{hjk} = G^i_{khj}.$$

In view of Euler's theorem and the equations (7.2) and (7.5), we have

(7.7) $$G^i_{jkh}y^j = G^i_{kjh}y^j = G^i_{khj}y^j = 0 .$$

As in case of Cartan covariant derivative, L. Berwald's covariant derivatives of metric function F, and the unit vector l^i and the vector y^i vanish identically, i.e.

(7.8) a) $\mathcal{B}_k F = 0$, b) $\mathcal{B}_k l^i = 0$ and c) $\mathcal{B}_k y^i = 0$.

But the Berwald covariant derivative of the metric tensor does not vanish and is given by

(7.9) $$\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h \mathcal{B}_h C_{ijk}.$$

The Berwald covariant differential operator commute with the partial differential operator with respect to y^i according to

(7.10) $$(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k)T^i_j = T^s_j G^i_{khs} - T^i_s G^s_{khj}$$

where T^i_j is an arbitrary tensor.

A Finsler space whose connection parameters G^i_{jk} are independent of y^i is called *affinely connected* or *Berwald space*. Thus the affinely connected or Berwald space is characterized by one of the equivalent conditions [120]

(7.11) a) $G^i_{jkh} = 0$ b) $C_{ijk|h} = 0$.

In particular, $\mathcal{B}_h g_{ij}$ vanishes for an affinely connected Finsler space. Berwald's connection parameters G^i_{jk} coincide with Cartan connection parameters Γ^{*i}_{jk} for a Landsberg space which is characterized by any one of the equivalent conditions

(7.12) $$y_s G^s_{jkh} = -2C_{jhk|l}\, y^l = -2P_{jkh} = 0\,.$$

Various authors denote the tensor $C_{ijk|h}\, y^h$ by P_{ijk} [27-32]. Since (7.11) imply (7.12), an affinely connected space is necessarily a Landsberg sapce. However, a Landsberg space need not be an affinely connected space.

8. Cartan Curvature Tensor

The commutation formulae for h-covariant derivative of an arbitrary vector field X^i are given by

(8.1) $$X^i_{|h|k} - X^i_{|k|h} = X^r K^i_{rhk} - (\dot{\partial}_r X^i) K^r_{shk}\, y^s$$

where

$$K^i_{rhk} := \partial_k \Gamma^{*i}_{hr} + (\dot{\partial}_l \Gamma^{*i}_{rk}) \Gamma^{*l}_{th}\, y^t + \Gamma^{*i}_{mk} \Gamma^{*m}_{hr} \qquad\qquad -k \mid h^{*}.$$

The tensor K^i_{rhk} defined above is called *Cartan's Curvature tensor*. This tensor is skew-symmetric in its last two lower indices h and k, i.e.

(8.2) $$K^i_{jhk} = -K^i_{jkh}\,,$$

and is positively homogeneous of degree zero in y^i.

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* $-k \mid h$ means the subtraction from the former term by interchanging the indices k and h.

The curvature tensor K^{i}_{jkh} satisfies the following identities known as *Bianchi identities*

(8.3) a) $$K^{i}_{jkh} + K^{i}_{hjk} + K^{i}_{khj} = 0$$

and

b) $$K^{r}_{ijh|k} + K^{r}_{ikj|h} + K^{r}_{ihk|j} + \{(\dot{\partial}_s \Gamma^{*r}_{ij}) K^{s}_{thk}$$

$$+ (\dot{\partial}_s \Gamma^{*r}_{ik}) K^{s}_{tjh} + (\dot{\partial}_s \Gamma^{*r}_{ih}) K^{s}_{tkj}\} y^{t} = 0 .$$

The associate tensor K_{ijhk} of K^{i}_{jkh} is given by

(8.4) a) $$K_{ijkh} := g_{rj} K^{r}_{ikh} .$$

The tensor K_{ijkh} also satisfies the condition

(8.4) b) $$K_{jihk} = -K_{ijhk} - 2C_{ijl} K^{l}_{rhk} y^{r} .$$

The tensor K^{i}_{jkh} also satisfies the following relations

(8.5) a) $$K^{i}_{jkh} y^{j} = H^{i}_{kh} ,$$

b) $$H^{i}_{jkh} = K^{i}_{jkh} + y^{m} (\dot{\partial}_j K^{i}_{mkh})$$

and

(8.5) c) $$H^{i}_{jkh} - K^{i}_{jkh} = P^{i}_{jk|h} + P^{r}_{jk} P^{i}_{rh} \qquad - k \mid h ,$$

where H^i_{jkh} is *Berwald's curvature tensor* to be defined in the next section i.e. Section 9.

We also have the following commutation formulae

(8.6) a) $$X^i|_k|_h - X^i|_h|_k = S^i_{jkh} X^j,$$

b) $$X^i_{|k}|_h - X^i|_{h|k} = P^i_{jkh} X^j - X^i|_j \, P^j_{kh} - X^i_{|j} C^j_{kh}$$

and

c) $$X^i_{|h|k} - X^i_{|k|h} = R^i_{jhk} X^j - X^i|_j K^j_{rhk} y^r$$

or

d) $$X^i_{|h|k} - X^i_{|k|h} = K^i_{jhk} X^j - X^i|_j K^j_{rhk} y^r,$$

where S^i_{jkh}, P^i_{jkh} and R^i_{jkh} are *v-curvature tensor, hv-curvature tensor* and *h-curvature tensor* respectively and are defined as follows:

(8.7) a) $$S^i_{jkh} := C^i_{kr} C^r_{jh} - C^i_{rh} C^r_{jk}.$$

b) $$P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm} P^m_{kh} - C^i_{jh|k}$$

and

c) $$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_l \Gamma^{*i}_{jk}) \Gamma^{*l}_{sh} y^s + C^i_{jm} (\partial_k \Gamma^{*m}_{sh} y^s$$
$$- \Gamma^{*m}_{kl} \Gamma^l_{sh} y^s) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} \qquad -k \mid h.$$

Tensor R^i_{jkh} satisfies the following identities

(8.8) a) $$R^r_{ijk|h} + R^r_{ikh|j} + R^r_{ihj|k} + y^m (R^l_{mkh} P^r_{ijl} + R^l_{mjk} P^r_{ihl} + R^l_{mhj} P^r_{ikl}) = 0,$$

b) $$R^i_{jkh} = K^i_{jkh} + C^i_{jm} H^m_{kh}$$

and

c) $$R^i_{jkh} y^j = K^i_{jkh} y^j = H^i_{kh}.$$

The associate tensor R_{ijhk} of R^r_{ihk} is given by

(8.9) a) $$R_{ijhk} = g_{jr} R^r_{ihk}$$

and satisfies

(8.9) b) $$R_{ijhk} = K_{ijhk} + C_{ijm} K^m_{rkh} y^r$$

which is skew-symmetric in the first two lower indices, i.e.,

(8.9) c) $R_{ijhk} = -R_{jihk}$.

The tensor P^i_{jkh} satisfies

(8.10) $$P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r.$$

The tensors H^i_{hk} and P^i_{kh} are *v(h)-torsion tensor* and (*v*) *hv-torsion tensor* respectively.

The tensor S^i_{jkh} is skew-symmetric in the last two lower indices, i.e.

(8.11) $$S^i_{jhk} = -S^i_{jkh} .$$

9. Berwald's Curvature Tensor

The commutation formula for Berwald's covariant differentiation is given as

$$(9.1) \qquad \mathcal{B}_k \mathcal{B}_h X^i - \mathcal{B}_h \mathcal{B}_k X^i = X^r H^i_{rhk} - (\dot{\partial}_r X^i) H^r_{hk}$$

where

$$(9.2) \qquad a) \qquad H^i_{rhk} := \partial_k G^i_{rh} + G^s_{rh} G^i_{sk} + G^i_{srk} G^s_h \qquad - k \mid h$$

and

$$b) \qquad H^r_{hk} := H^r_{shk} y^s .$$

The formula (9.1) is called the *Generalized Ricci Commutation formula.* The tensor H^i_{rhk}, defined above is called *Berwald's curvature tensor.* This tensor is skew-symmetric in its last two lower indices i.e. h and k and positively homogeneous of degree zero in y^i. The tensor H^r_{hk} is skew-symmetric in its lower indices and positively homogeneous of degree one in y^i. The curvature tensor H^i_{rhk} and the tensor H^i_{hk} are also related by

$$(9.3) \qquad \partial_r H^i_{hk} = H^i_{rhk} .$$

Berwald defined a tensor H^i_k, called by him as *deviation tensor* by

$$(9.4) \qquad H^i_k := 2\partial_k G^i - \partial_s G^i_k y^s + 2G^i_{ks} G^s - G^i_s G^s_k .$$

The deviation tensor is positively homogeneous of degree two in y^i.

Berwald constructed the tensors H^i_{hk} and H^i_{rhk} from the deviation tensor H^i_k as follows

(9.5) a) $$H^i_{hk} = \frac{1}{3}(\dot{\partial}_h H^i_k - \dot{\partial}_k H^i_h)$$

and

b) $$H^i_{rhk} = \frac{1}{3}\dot{\partial}_r(\dot{\partial}_h H^i_k - \dot{\partial}_k H^i_h).$$

Contracting the indices i and k in equation (9.4) and (9.5), we get the following

(9.6) a) $$H = H^i_i/(n-1),$$

b) $$H_h = H^i_{hi}$$

and

c) $$H_{rh} = H^i_{rhi}.$$

Since the contraction of the indices does not change the homogeneity in y^i, the degree of homogeneity of the tensors H_{rh}, H_h scalar H in y^i are zero, one and two respectively.

In view of Euler's theorem on homogeneous function we have the following :

(9.7) a) $$y^r \dot{\partial}_r H^i_{jkh} = y^r \dot{\partial}_j \dot{\partial}_r H^i_{kh} = 0,$$

b) $$H^i_{jk} y^j = -H^i_{kj} y^j = H^i_k$$

and

c) $$H^i_j y^j = 0.$$

The tensors H_{kh}, H_k and the scalar H are also connected by

(9.8) a) $$H_{kh} = \dot{\partial}_k H_h ,$$

b) $$H_{kh} y^k = H_h$$

and

c) $$H_k y^k = (n-1)H .$$

The *Bianchi identities* for Berwald curvature tensor are given by

(9.9) a) $$H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0 ,$$

b) $$\mathcal{B}_m H^i_{jkh} + \mathcal{B}_h H^i_{jmk} + \mathcal{B}_k H^i_{jhm} + H^r_{kh} G^i_{mjr} + H^r_{mk} G^i_{hjr} + H^r_{hm} G^i_{kjr} = 0$$

and

c) $$\mathcal{B}_m H^i_{kh} + \mathcal{B}_h H^i_{mk} + \mathcal{B}_k H^i_{hm} = 0 .$$

Contraction of the indices i and j in (9.9a) and utilization of (9.6c) and the skew-symmetric property of the curvature tensor H^i_{jkh} in its last two lower indices give

(9.10) $$H^r_{rkh} = H_{hk} - H_{kh} .$$

Another important identity connecting the Berwald curvature tensor and the tensor G^i_{jkh} is given by [120]

(9.11) $$\mathcal{B}_l G^i_{jkh} - \mathcal{B}_k G^i_{ljh} = \dot{\partial}_h H^i_{jkl} .$$

The above tensors also satisfy the following

(9.12) a) $$y_t H^t_{kh} = 0,$$

b) $$y_t H^i_h = 0$$

and

c) $$g_{tk} H^t_h = g_{th} H^i_k .$$

The tensor $H_{jk.h}$, defined by

(9.13) $$H_{jk.h} := g_{ik} H^t_{jh} ,$$

satisfies

(9.14) $$H_{[jk\,h]} = 0 .$$

10. Projective Curvature Tensor

Suppose that $F_n = (X_n, F)$ and $\overline{F}_n = (X_n, \overline{F})$ be two Finsler spaces on a common underlying space X_n. Let us consider a transformation $F_n \to \overline{F}_n$ which carries every geodesic of F_n to a geodesic of $\overline{F}_n$ and the inverse is also true. This transformation is called *projective*.

The equations of a geodesic on F_n are given by

(10.1) $$\frac{dy^i}{dt} + 2G^i(x^k, y^k) = \tau y^i$$

where

$$\tau = \frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}}. \tag{10.2}$$

The differential equations of a geodesic may also be written as

$$[y^i + 2G^i(x, y)]y^k = [y^k + 2G^k[x, y)]y^i \ . \tag{10.3}$$

If the function $G^i(x, y)$ are replaced by new functions $\overline{G}^i(x, y)$ defined by

$$\overline{G}^i(x, y) := G^i(x, y) - P(x, y)y^i \tag{10.4}$$

then the equations (10.3) remain unchanged. $P(x, y)$ in the equation (10.4) is an arbitrary scalar function which is positively homogeneous of degree one in y^i. We shall call (10.4) as *projective change of functions* G^i.

Differentiating (10.4) partially with respect to y^h and applying (6.3d), we get

$$\overline{G}^i_h = G^i_h - y^i P_h - \delta^i_h P \ . \tag{10.5}$$

Under the projective change (10.4), the Berwald connection parameters G^i_{jk} are transformed according to

$$\overline{G}^i_{jk} = G^i_{jk} - \delta^i_j P_k - \delta^i_k P_j - y^i P_{jk} , \tag{10.6}$$

where P_k and P_{jk} are the directional derivatives of P, i.e.

(10.7) a) $P_h = \dot{\partial}_h P$, b) $P_{kh} = \dot{\partial}_k \dot{\partial}_h P$.

P_k and P_{jk} satisfy the following

(10.8) a) $P_k y^k = P$, b) $P_{jk} y^k = 0$.

L. Berwald [8, 12] deduced a tensor

(10.9) $$W^i_h := H^i_h - H\delta^i_h - \frac{1}{n+1}(\partial_r H^r_h - \dot{\partial}_h H)y^i$$

which remains invariant under the projective change (10.4). This tensor is called *projective deviation tensor*. In view of the homogeneity of H^i_j and H in y^i, the tensor W^i_h defined by (10.9) is also positively homogeneous of degree two in y^i and satisfies

(10.10) a) $$W^i_i = 0,$$

b) $$W^i_j y^j = 0$$

and

c) $$(\dot{\partial}_h W^i_j)y^j = -W^i_h .$$

Analogous to Berwald's tensors, the following projective tensors have been defined;

(10.11) a) $$W^i_{jk} := \frac{1}{3}(\dot{\partial}_j W^i_k - \dot{\partial}_k W^i_j)$$

and

b) $$W^i_{hjk} := \dot{\partial}_h W^i_{jk} = \frac{1}{3}\dot{\partial}_h(\dot{\partial}_j W^i_k - \dot{\partial}_k W^i_j)$$

which also remain invariant under the projective change (10.4) and are skew-symmetric in the lower indices j and k. The tensor W^i_{hjk} is called the *generalized Weyl's projective curvature tensor*.

The partial differentiations of (10.9) with respect to directional arguments, in view of equations (9.5) and (10.11), give explicit expressions for the above tensros.

(10.12) a) $$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H^r_{rkj} + \{ \frac{\delta^i_j}{n^2 - 1} (nH_k + y^r H_{kr}) - j \mid k \}$$

together with

(10.12) b) $$W^i_{hjk} = H^i_{hjk} + \frac{1}{n+1} (H^r_{rkj} \delta^i_h + y^i \dot{\partial}_h H^r_{rkj})$$

$$+ \frac{1}{n^2 - 1} \{ (n \dot{\partial}_h H_k \delta^i_j + H_{kh} \delta^i_j + y^r \dot{\partial}_h H_{kr} \delta^i_j) - j / k \} .$$

The tensor W^i_{hjk} is the generalization of the Weyl's projective curvature tensor and satisfies the following identities

(10.13) a) $$W^i_{jkh} y^j = W^i_{kh}$$

and

b) $$W^i_{jk} y^j = W^i_k .$$

Contracting W^i_{jkh} with respect to i and j, we get

(10.14) a) $$W^i_{ikh} = W^i_{kih} = W^i_{khi} = 0 ,$$

b) $$W^i_{ik} = -W^i_{ki} = 0 .$$

11. Lie-Differentiation

Let $v^i(x^j)$ be a contravariant vector field independent of directional arguments defined over a Finsler space F_n. Let us consider a transformation

(11.1) $$\bar{x}^i = x^i + \in v^i(x)$$

where $\in$ is an infinitesimal constant. The corresponding variation in y^i is represented by

(11.2) $$\bar{y}^i = y^i + \in (\partial_j v^i) y^j .$$

The transformation represented by equation (11.1) is called an *infinitesimal transformation*.

The transformation (11.1) gives rise to a process of differentiation called *Lie-differentiation.*

Let X^i be an arbitrary contravariant vector field. Its Lie-derivative with respect to the above infinitesimal transformation is given by

(11.3) $$£ X^i = X^i_{|r} v^r - X^r v^i_{|r} + (\dot{\partial}_r X^i) v^r_{|s} y^s ,$$

where symbol £ stands for the Lie-differentiation.

In view of equation (11.3) the Lie-derivatives of y^i and v^i with respect to the above infinitesimal transformation vanish.

Let T^i_{kh} be an arbitrary tensor field. Its Lie-derivative with respect to the above infinitesimal transformation is given by

(11.4) a) $$£ T^i_{kh} = T^i_{kh|r} v^r - T^r_{kh} v^i_{|r} + T^i_{rh} v^r_{|k} + T^i_{kr} v^r_{|h} + (\dot{\partial}_r T^i_{kh}) v^r_{|s} y^s .$$

In terms of Berwald's covariant derivative the same expression may be written as

(11.4) b) $$£ T^i_{kh} = v^r \mathcal{B}_r T^i_{kh} - T^r_{kh} \mathcal{B}_r v^i + T^i_{rh} \mathcal{B}_k v^r + T^i_{kr} \mathcal{B}_h v^r + (\dot{\partial}_r T^i_{kh}) \mathcal{B}_s v^r y^s .$$

On the other hand the Lie-derivative of the connection parameters Γ^{*i}_{jk} and G^i_{kh} are given by [19, 120, 146, 147]

(11.5) a) $$\pounds\Gamma^{*i}_{kh} = v^i_{|k|h} + K^i_{khr}v^r + (\dot\partial_r \Gamma^{*i}_{kh})v^r_{|s}y^s$$

and

b) $$\pounds G^i_{kh} = \mathcal{B}_k\mathcal{B}_h v^i + H^i_{khr}v^r + (\dot\partial_r G^i_{kh})\,\mathcal{B}_s v^r y^s .$$

The process of Lie-differentiation commutes with the processes of partial and covariant differentiations according to

(11.6) a) $$(\partial_j \pounds - \pounds\, \dot\partial_j)\Omega = 0 ,$$

b) $$\pounds(X^i_{|k}) - (\pounds X^i)_{|k} = X^r \pounds\, \Gamma^{*i}_{rk} - (\dot\partial_r X^i)\pounds\, G^r_k$$

and

c) $$(\pounds\mathcal{B}_k - \mathcal{B}_k\pounds)\; X^i = X^r \pounds\; G^i_{rk} - (\dot\partial_r X^i)\pounds\; G^r_k$$

where Ω is any geometric object such as scalar, vector or connection coefficients.

The infinitesimal transformation (11.1) defines a motion, affine motion, projective motion or conformal motion if it preserves the distance between two points, parallelism of pair of vectors, the geodesic or the angle between pairs of vectors respectively. Necessary and sufficient conditions for the transformation (11.1) to be a motion, affine motion, projective motion and conformal motion are respectively given by

(11.7) a) $$\pounds g_{kh} = 0$$

b) (i) $$\pounds\; \Gamma^{*i}_{kh} = 0$$

(ii) $$\pounds\, G^i_{kh} = 0$$

c) $$\pounds\, G^i_{kh} = \delta^i_k P_h + \delta^i_h P_k + y^i P_{kh}$$

and

d) $$\pounds\, g_{kh} = \phi, \phi \text{ is a scalar point function,}$$

where P_h and P_{kh} are defined as (10.7), P being a scalar positively homogeneous of degree one in y^i and ϕ is a function of x^i only, i.e. $\phi = \phi(x^i)$.

It is well known that every motion is an affine motion and every affine motion is a projective motion. A projective motion need not be an affine motion. A projective motion which is not an affine motion will be called as *non-affine projective motion.*

* * * * *

Chapter II

ON GENERALIZED R^h-RECURRENT SPACE

1. Introduction

A three dimensional Riemannian space having recurrent curvature (the covariant derivative of whose curvature tensor is expressible as tensor product of a non-null covariant vector field and the curvature tensor itself) was first introduced by H. S. Ruse [122]. This theory of recurrent curvature was extended to n-dimensional Riemannian and non-Riemannian spaces by A. G. Walker [139]. Since then a large number of differential geometers including Y. C. Wong [140-142], Y. C. Wong and K. Yano [143], K. Takano [133, 134, 137], S. Yamaguchi [145], T. Adati and T. Miyazawa [2, 3], T. Miyazawa [66, 67], W. M. Yang and Y. T. Liu [146] discussed the theory of such spaces and the recurrence of projective and conformal curvature tensors.

For the first time, A. Moór [70, 73, 75] extended this concept of recurrence curvature to a Finsler space. Because of different connections of a Finsler space, we have different curvature tensors of a Finsler space, e.g. Berwald curvature tensor H^i_{jkh}, Cartan curvature tensors $K^i_{jkh}, R^i_{jkh}, P^i_{jkh}$ and S^i_{jkh}. The recurrence of different curvature tensors have been discussed by R. N. Sen [123], R. B. Misra and F. M. Meher [63], R. S. Mishra and H. D. Pande [65], P. N. Pandey and R. B. Misra [112], B. B. Sinha and S. P. Singh [127], R. S. Sinha [128, 129], P. N. Pandey [83, 84, 87-89, 94, 95, 97, 99-104] , R. S. D. Dubey and A. K. Srivastava [23], V. J. Dwivedi [24] , Reema Verma [138], P. N. Pandey and Shalini Dikshit [107], P. N. Pandey and Reema Verma [113-115] and others. Shalini Dikshit [21] discussed a Finsler space having birecurrent Berwald curvature tensor. Fahmi Yaseen Abdo Qasem [149] generalized an R^h-birecurrent sapce in which Cartan's third curvature tensor satisfies the generalized birecurrence conditions with respect to Cartan's connection Γ^{*i}_{jk} and discussed various properties of such spaces.

In this chapter we shall consider an R^h-recurrent space in which Cartan's third curvature tensor satisfies the generalized-recurrence condition with respect to Cartan's connection Γ^{*i}_{jk}. The conditions of recurrence, birecurrence and generalized birecurrence are particular cases of the generalized recurrence condition. The aim of this chapter is to discuss various properties of such space.

2. R^h-Generalized Recurrent Space

A Finsler space whose third curvature tensor R^i_{jkh} of E. Cartan satisfies the recurrence property with respect to Cartan's connection Γ^{*i}_{jk} was discussed by Reema Verma [138] and called by her as an R^h*-recurrent space*. Thus, an R^h-recurrent space is characterized by the condition

$$R^i_{jkh|m} = \lambda_m R^i_{jkh},\ R^i_{jkh} \neq 0. \tag{2.1}$$

The non-zero covariant vector field λ_m is the recurrence vector field.

A more general Finsler space in which Cartan third curvature tensor satisfies the birecurrence condition with respect to Cartan's connection Γ^{*i}_{jk} was discussed by Shalini Dikshit [21] and called by her an R^h*-birecurrent space*. Thus, an R^h-birecurrent space is characterized by

$$R^i_{jkh|m|l} = a_{lm} R^i_{jkh},\quad R^i_{jkh} \neq 0 \tag{2.2}$$

where a_{lm} is non-zero covariant tensor field of order 2, called as recurrence tensor field. She also proved there in that an R^h-recurrent space is an R^h-birecurrent but its converse need not be true.

Fahmi Yaseen Abdo Qasem [149] discussed a Finsler space for which Cartan third curvature tensor satisfies the following conditions:

(2.3) a) $$R^i_{jkh|m|l} = \lambda_l R^i_{jkh|m} + a_{lm} R^i_{jkh},$$

b) $$R^i_{jkh|m|l} = \lambda_m R^i_{jkh|l} + a_{lm} R^i_{jkh},$$

c) $$R^i_{jkh|m|l} = \lambda_l R^i_{jkh|m}$$

and

d) $$R^i_{jkh|m|l} = \lambda_m R^i_{jkh|l},$$

where λ_l and a_{lm} are non-zero covariant vector field and covariant tensor field of order one and two respectively. The space satisfying the conditions (2.3a), (2.3b), (2.3c) and (2.3d) have been called as *R^h-generalized birecurrent space of first kind, R^h-generalized birecurrent space of the second kind, special R^h-generalized birecurrent space of the first kind* and *special R^h-generalized birecurrent space of the second kind* respectively. He denoted them by R^h-$GBR1$-F_n, R^h-$GBR2$-F_n, R^h-$SGBR1$-F_n and R^h-$SGBR2$-F_n respectively.

In this chapter, we propose to study a Finsler space whose particular cases are these spaces .

Let us consider a Finsler space whose Cartan third curvature tensor satisfy

(2.4) $$\alpha R^i_{jkh|m|l} + \beta_l R^i_{jkh|m} + \gamma_m R^i_{jkh|l} + \nu_{lm} R^i_{jkh} = 0, \quad R^i_{jkh} \neq 0$$

where α is a scalar, β_l and γ_m are covariant vector fields and ν_{lm} is a covariant tensor field of rank 2. The space satisfying the condition (2.4) will be called an R^h-generalized recurrent space. We shall denote it briefly by R^h-GR-F_n.

If we take $\alpha = 0$ and $\beta_l = 0$ in (2.4), this condition reduces to (2.1) which is the condition satisfied by the curvature tensor of a recurrent space. Thus $\alpha = 0$ and $\beta_l = 0$ reduces an R^h-*GR*-F_n *into* R^h-recurrent space. Similarly $\gamma_m = 0$ and $\beta_l = 0$ reduces an R^h-*GR*-F_n into an R^h-birecurrent space, $\gamma_m = 0$ reduces an R^h-*GR*-F_n into an R^h-*GBR1*-F_n, $\beta_l = 0$ reduces an R^h-*GR*-F_n into R^h-*GBR2*-F_n, $v_{lm=0}$ and $\gamma_m = 0$ reduces an R^h-*GR*-F_n into an R^h-*SGBR1*-F_n and $v_{lm=0}$ and $\beta_l = 0$ reduces an R^h-*GR*-F_n into an R^h-*SGBR2*-F_n.

Since the metric tensor g_{ij} of a Finsler space is a covariant constant with respect to Cartan's connection Γ^{*i}_{jk}, the transvection of (2.4) by the metric tensor g_{ip} and the application of (1.6.10a) and (1.8.9a) yield

$$\alpha R_{jpkh|m|l} + \beta_l R_{jpkh|m} + \gamma_m R_{jpkh|l} + v_{lm} R_{jpkh} = 0 . \tag{2.5}$$

Conversely, the transvection of (2.5) by the associate tensor g^{ip} of the metric tensor g_{ij} yield (2.4). Thus, the condition (2.4) is equivalent to the condition (2.5). Therefore an R^h-generalized recurrent space may be characterized by the condition (2.5).

Contracting the indices i and h in (2.4), we get

$$\alpha R_{jk|m|l} + \beta_l R_{jk|m} + \gamma_m R_{jk|l} + v_{lm} R_{jk} = 0 . \tag{2.6}$$

Thus the Ricci tensor R_{jk} [16] of an R^h-*GR*-F_n satisfy (2.6).

Conversely, if the Ricci tensor of a Finsler space satisfies (2.6) then it need not be an R^h-*GR*-F_n. However, the converse is also true if the dimension of the Finsler space is three or the space is *R3*-like [40]. This may be proved as follows:

We know that the curvature tensor R_{ijkh} of a three dimensional Finsler space is of the form [43]

(2.7) a) $$R_{ijkh} = g_{ik}L_{jh} + g_{jh}L_{ik} \qquad -k/h$$

where

b) $$L_{ik} = \frac{1}{n-2}(R_{ik} - \frac{r}{2}g_{ik})$$

and

c) $$r = \frac{1}{n-1}R_i^i \ .$$

Transvecting (2.6) by g^{jp}, we get

(2.8) $$\alpha R^p_{k|m|l} + \beta_l R^p_{k|m} + \gamma_m R^p_{k|l} + \nu_{lm} R^p_k = 0 \ .$$

Contracting the indices p and k in (2.8) and using (2.7c), we get

(2.9) $$\alpha r_{|m|l} + \beta_l r_{|m} + \gamma_m r_{|l} + \nu_{lm} r = 0 \, .$$

In view of (2.7b), the second covariant derivative of L_{ik} in the sense of Cartan gives

$$L_{ik|m|l} = \frac{1}{n-2}(R_{ik|m|l} - \frac{r_{|m|l}}{2}g_{ik}) \, .$$

From equation (2.6) and (2.9), the above equation may be written as

$$\alpha L_{ik|m|l} = \frac{1}{n-2}[-\beta_l R_{ik|m} - \gamma_m R_{ik|l} - \nu_{lm} R_{ik} + \frac{1}{2}(\beta_l r_{|m} + \gamma_m r_{|l} + \nu_{lm} r) g_{ik}] \, .$$

Above equation may be written as

(2.10) $$\alpha L_{ik|m|l} + \beta_l L_{ik|m} + \gamma_m L_{ik|l} + \nu_{lm} L_{ik} = 0 \ .$$

Differentiating (2.7a) covariantly twice with respect to x^m and x^l successively in the sense of Cartan and using the equation (2.10) and (1.8.9a), we have

(2.11) $$\alpha R^i_{jkh|l|m|l} + \beta_l R^i_{jkh|m} + \gamma_m R^i_{jkh|l} + v_{lm} R^i_{jkh} = 0 .$$

This equation shows that a three dimensional Ricci-GR-F_n is necessarily R^h-GR-F_n. This leads to

Theorem 2.1. *An R^h-GR-F_n is Ricci-GR-F_n, but converse need not be true. However, if the space is R3-like then the converse is also true.*

If we take $\alpha = 0$ and $\beta_l = 0$, equation (2.6) reduces to

$$\gamma_m R_{jk|l} + v_{lm} R_{jk} = 0 ,$$

which may be written as

(2.12) $$R_{jk|l} = a_l R_{jk} ,$$

where

$$a_l = \frac{-v_{lm}}{\gamma_m} .$$

Also putting $\alpha = 0$ and $\beta_l = 0$ in the equation (2.11), we get

$$\gamma_m R^i_{jkh|l} + v_{lm} R^i_{jkh} = 0 ,$$

which may be written as

(2.13) $$R^i_{jkh|l} = a_l R^i_{jkh} .$$

Thus from the equations (2.12) and (2.13), we see that $\alpha = 0$ and $\beta_l = 0$ reduce theorem 2.1 to the following

Corollary 2.1. *An R^h-recurrent Finsler space is Ricci-recurrent but converse need not be true. However, if the space is R3-like then the converse is also true. This corollary has been proved by Reema Verma* [138].

Similarly $\gamma_m = 0$ and $\beta_l = 0$ reduce theorem 2.1 to

Corollary 2.2. *An R^h-birecurrent space is Ricci-birecurrent but converse need not be true. However, if the space is R3-like then the converse is also true.*

$\gamma_m = 0$, *reduce theorem* 2.1 *to*

Corollary 2.3. *An R^h-GBR1-F_n is Ricci-GBR1-F_n but converse need not be true. However, if the space is R3-like then the converse is also true.*

$\beta_l = 0$ *reduce theorem* 2.1 *to*

Corollary 2.4. *An R^h-GBR2-F_n is Ricci-GBR2-F_n but converse need not be true. However, if the space is R3-like then the converse is also true.*

$v_{lm} = 0$ *and* $\gamma_m = 0$ *reduce theorem* 2.1 *to*

Corollary 2.5. *An R^h-SGBR1-F_n is Ricci-SGBR1-F_n but converse need not be true. However, if the space is R3-like then the converse is also true.*

$v_{lm} = 0$ *and* $\beta_l = 0$ *reduce theorem* 2.1 *to*

Corollary 2.6. *An R^h-SGBR2-F_n is Ricci-SGBR2-F_n but converse need not be true. However, if the space is R3-like then the converse is also true.*

Thus from the above corollaries, we see that the theorems proved by the above authors may be derived from theorem 2.1 as the particular cases.

Transvecting (2.4) by y^j and using (1.8.8c), we get

$$\alpha H^i_{kh|m|l} + \beta_l H^i_{kh|m} + \gamma_m H^i_{kh|l} + v_{lm} H^i_{kh} = 0. \tag{2.14}$$

Transvecting (2.14) by y^k and using (1.9.7b), we get

(2.15) $$\alpha H^i_{hlm|l} + \beta_l H^i_{hlm} + \gamma_m H^i_{hl|l} + v_{lm} H^i_h = 0 .$$

Contracting the indices i and h in (2.14) and using (1.9.6b), we get

(2.16) $$\alpha H_{klm|l} + \beta_l H_{klm} + \gamma_m H_{kl|l} + v_{lm} H_k = 0 .$$

Contracting the indices i an h in (2.15) and using (1.9.6a), we have

(2.17) $$\alpha H_{lm|l} + \beta_l H_{lm} + \gamma_m H_{l|l} + v_{lm} H = 0 .$$

Thus, we may conclude

Theorem 2.2. *The tensors H^i_{kh}, H^i_h, the vector H_k and the scalar H of an R^h-GR-F_n are h-GR-F_n.*

If we take particular cases i.e. (i) $\alpha = 0$, $\beta_l = 0$ (ii) $\gamma_m = 0, \beta_l = 0$, (iii) $\gamma_m = 0$, (iv) $\beta_l = 0$, (v) $v_{lm} = 0, \gamma_m = 0$ and (vi) $v_{lm} = 0, \beta_l = 0$, theorem 2.2 reduces to the following

Corollary 2.7.

(i) *The tensors H^i_{kh}, H^i_h, the vector H_k and the scalar H of an R^h-recurrent are h-recurrent.*

(ii) *The tensors H^i_{kh}, H^i_h, the vector H_k and the scalar H of an R^h-birecurrent are h-birecurrent.*

(iii) *The tensors H^i_{kh}, H^i_h, the vector H_k and the scalar H of an R^h-GBR1-F_n are h-GBR1.*

(iv) *The tensors H^i_{kh}, H^i_h, the vector H_k and the scalar H of an R^h-GBR2-F_n are h-GBR2.*

(v) *The tensors H^i_{kh}, H^i_h, the vector H_k and the scalar H of an R^h-SGBR1-F_n are h-SGBR1.*

and

(vi) *The tensors* H^i_{kh}, H^i_h, *the vector* H_k *and the scalar* H *of an* R^h*-SGBR2-*F_n *are h-SGBR2.*

Now we shall try to find the necessary and sufficient condition for Berwald curvature tensor H^i_{jkh} to be *h-GR*.

Differentiating (2.14) partially with respect to y^j and using (1.9.3), we have

$$\alpha \dot\partial_j H^i_{kh|m|l} + \beta_l \dot\partial H^i_{kh|m} + \gamma_m \dot\partial_j H^i_{kh|l} + \nu_{lm} H^i_{kh} \tag{2.18}$$

$$+(\dot\partial_j \alpha) H^i_{kh|m|l} + (\dot\partial_j \beta_l) H^i_{kh|m} + (\dot\partial_j \gamma_m) H^i_{kh|l} + (\dot\partial_j \nu_{lm}) H^i_{kh} = 0 .$$

Using the commutation formula exhibited by (1.6.12b), we get

$$\alpha[(\dot\partial_j H^i_{kh|m})_{|l} + H^i_{kh|m} \dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh|m} \dot\partial_j \Gamma^{*r}_{kl} \tag{2.19}$$

$$- H^i_{kr|m} \dot\partial_j \Gamma^{*r}_{hl} - H^i_{kh|r} \dot\partial_j \Gamma^{*r}_{ml} - \dot\partial_r H^i_{kh|m} P^r_{jl}$$

$$+ \beta_l [H^i_{jkh|m} + H^r_{kh} \dot\partial_j \Gamma^{*i}_{rm} - H^i_{rh} \dot\partial_j \Gamma^{*r}_{km} - H^i_{kr} \dot\partial_j \Gamma^{*r}_{hm} - H^i_{rkh} P^r_{jm}]$$

$$+ \gamma_m [H^i_{jkh|l} + H^r_{kh} \dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh} \dot\partial_j \Gamma^{*r}_{kl} - H^i_{kr} \dot\partial_j \Gamma^{*r}_{hl} - H^i_{rkh} P^r_{jl}]$$

$$+ \nu_{lm} H^i_{jkh} + (\dot\partial_j \alpha) H^i_{kh|m|l} + (\dot\partial_j \beta_l) H^i_{kh|m} + (\dot\partial_j \gamma_m) H^i_{kh|l} (\dot\partial_j \nu_{lm}) H^i_{kh} = 0 .$$

Again applying the commutation formula (1.6.12b), we have

$$\alpha[(H^i_{jkh|m} + H^r_{kh} \dot\partial_j \Gamma^{*i}_{rm} - H^i_{rh} \dot\partial_j \Gamma^{*r}_{km} - H^i_{kr} \dot\partial_j \Gamma^{*r}_{hm} - H^i_{rkh} P^r_{jm})_{|l} \tag{2.20}$$

$$+ H^r_{kh|m} \dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh|m} \dot\partial_j \Gamma^{*r}_{kl} - H^i_{kr|m} \dot\partial_j \Gamma^{*r}_{hl} - H^i_{kh|r} \dot\partial_j \Gamma^{*r}_{ml}$$

$$-H^{i}_{rkh|lm}P^{r}_{jl} - H^{s}_{kh}\dot{\partial}_{r}\Gamma^{*i}_{sm}P^{r}_{jl} + H^{i}_{sh}\dot{\partial}_{r}\Gamma^{*s}_{km}P^{r}_{jl} + H^{i}_{ks}\dot{\partial}_{r}\Gamma^{*s}_{hm}P^{r}_{jl}$$

$$+H^{i}_{skh}P^{s}_{rm}P^{r}_{jl}] + \beta_{l}[H^{i}_{jkh|m} + H^{r}_{kh}\dot{\partial}_{j}\Gamma^{*i}_{rm} - H^{i}_{rh}\dot{\partial}_{j}\Gamma^{*r}_{km}$$

$$-H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{hm} - H^{i}_{rkh}P^{r}_{jm}] + \gamma_{m}[H^{i}_{jkh|l} + H^{r}_{kh}\dot{\partial}_{j}\Gamma^{*i}_{rl}$$

$$-H^{i}_{rh}\dot{\partial}_{j}\Gamma^{*r}_{kl} - H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{hl} - H^{i}_{rkh}P^{r}_{jl}] + \nu_{lm}H^{i}_{jkh}$$

$$+(\dot{\partial}_{j}\alpha)H^{i}_{kh|m|l} + (\dot{\partial}_{j}\beta_{l})H^{i}_{kh|m} + (\dot{\partial}_{j}\gamma_{m})H^{i}_{kh|l} + (\dot{\partial}_{j}\nu_{lm})H^{i}_{kh} = 0.$$

which may be rewritten as

(2.21) $$\alpha H^{i}_{jkh|m|l} + \beta_{l}H^{i}_{jkh|m} + \gamma_{m}H^{i}_{jkh|l} + \nu_{lm}H^{i}_{jkh} = -\alpha[H^{r}_{kh}$$

$$\dot{\partial}_{j}\Gamma^{*i}_{rm} - H^{i}_{rh}\dot{\partial}_{j}\Gamma^{*r}_{km} - H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{hm} - H^{i}_{rkh}P^{r}_{jm}]_{|l}$$

$$-\alpha[H^{r}_{kh|m}\dot{\partial}_{j}\Gamma^{*i}_{rl} - H^{i}_{rh|m}\dot{\partial}_{j}\Gamma^{*r}_{kl} - H^{i}_{kr|m}\dot{\partial}_{j}\Gamma^{*r}_{hl}$$

$$-H^{i}_{kh|r}\dot{\partial}_{j}\Gamma^{*r}_{ml} - H^{i}_{rkh|m}P^{r}_{jl} - H^{s}_{kh}\dot{\partial}_{r}\Gamma^{*i}_{sm}P^{r}_{jl} + H^{i}_{sh}\dot{\partial}_{r}\Gamma^{*s}_{km}P^{r}_{jl}$$

$$+H^{i}_{ks}\dot{\partial}_{r}\Gamma^{*s}_{hm}P^{r}_{jl} + H^{i}_{skh}P^{s}_{rm}P^{r}_{jl}] - \beta_{l}[H^{r}_{kh}\dot{\partial}_{j}\Gamma^{*i}_{rm}$$

$$-H^{i}_{rh}\dot{\partial}_{j}\Gamma^{*r}_{km} - H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{hm} - H^{i}_{rkh}P^{r}_{jm}] - \gamma_{m}[H^{r}_{kh}\dot{\partial}_{j}\Gamma^{*i}_{rl}$$

$$-H^{i}_{rh}\dot{\partial}_{j}\Gamma^{*r}_{kl} - H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{hl} - H^{i}_{rkh}P^{r}_{jl}] - (\dot{\partial}_{j}\alpha)H^{i}_{kh|m|l}$$

$$-(\dot{\partial}_{j}\beta_{l})H^{i}_{kh|m} - (\dot{\partial}_{j}\gamma_{m})H^{i}_{kh|l} - (\dot{\partial}_{j}\nu_{lm})H^{i}_{kh}.$$

The above equation shows that

(2.22) $$\alpha H^i_{jkh|m|l} + \beta_l H^i_{jkh|m} + \gamma_m H^i_{jkh|l} + v_{lm} H^i_{jkh} = 0$$

if and only if

(2.23) $$\alpha[(H^r_{kh}\dot\partial_j \Gamma^{*i}_{rm} - H^i_{rh}\dot\partial_j \Gamma^{*r}_{km} - H^i_{kr}\dot\partial_j \Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm})_{|l}$$

$$+ H^r_{kh|m}\dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh|m}\dot\partial_j \Gamma^{*r}_{kl} - H^i_{kr|m}\dot\partial_j \Gamma^{*r}_{hl} - H^i_{kh|r}\dot\partial_j \Gamma^{*r}_{ml}$$

$$- H^i_{rkh|m}P^r_{jl} - H^s_{kh}\dot\partial_r \Gamma^{*i}_{sm}P^r_{jl} + H^i_{sh}\dot\partial_r \Gamma^{*s}_{km}P^r_{jl}$$

$$+ H^i_{ks}\dot\partial_r \Gamma^{*s}_{hm}P^r_{jl} + H^i_{skh}P^s_{rm}P^r_{jl}] + \beta_l[H^r_{kh}\dot\partial_j \Gamma^{*i}_{rm}$$

$$- H^i_{rh}\dot\partial_j \Gamma^{*r}_{km} - H^i_{kr}\dot\partial_j \Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm}] + \gamma_m[H^r_{kh}\dot\partial_j \Gamma^{*i}_{rl}$$

$$- H^i_{rh}\dot\partial_j \Gamma^{*r}_{kl} - H^i_{kr}\dot\partial_j \Gamma^{*r}_{hl} - H^i_{rkh}P^r_{jl}] + (\dot\partial_j \alpha)H^i_{kh|m|l}$$

$$+ (\dot\partial_j \beta_l)H^i_{kh|m} + (\dot\partial_j \gamma_m)H^i_{kh|l} + (\dot\partial_j v_{lm})H^i_{kh} = 0.$$

Thus, we have

Theorem 2.3. *The Berwald curvature tensor* H^i_{jkh} *of an* R^h*-GR-*F_n *is h-GR-*F_n *if and only if* (2.23) *holds good.*

Now if we take $\alpha = 0$ and $\beta_l = 0$ in (2.21) we have

(2.24) $$\gamma_m H^i_{jkh|l} + v_{lm} H^i_{jkh} = -\gamma_m[H^r_{kh}\dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh}\dot\partial_j \Gamma^{*r}_{kl}$$

$$- H^i_{kr}\dot\partial_j \Gamma^{*r}_{hl} - H^i_{rkh}P^r_{jl}] - (\dot\partial_j \gamma_m)H^i_{kh|l} - (\dot\partial_j v_{lm})H^i_{kh}\ .$$

This equation may be written as

$$H^i_{jkh|l} + \frac{v_{lm}}{\gamma_m} H^i_{jkh} = -[H^r_{kh} \dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh} \dot\partial_j \Gamma^{*r}_{kl}$$

$$- H^i_{kr} \dot\partial_j \Gamma^{*r}_{hl} - H^i_{rkh} P^r_{jl}] - \frac{1}{\gamma_m} (\dot\partial_j \gamma_m) H^i_{kh|l} - \frac{1}{\gamma_m} (\dot\partial_j v_{lm}) H^i_{kh} \, .$$

or

$$(2.25) \qquad H^i_{jkh|l} + \frac{v_{lm}}{\gamma_m} H^i_{jkh} = -[H^r_{kh} \dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh} \dot\partial_j \Gamma^{*r}_{kl}$$

$$- H^i_{kr} \dot\partial_j \Gamma^{*r}_{hl} - H^i_{rkh} P^r_{jl}] - \dot\partial_j (\gamma_m \cdot \frac{1}{\gamma_m}) H^i_{kh|l} + \gamma_m (\dot\partial_j \frac{1}{\gamma_m})$$

$$H^i_{kh|l} - \dot\partial_j (v_{lm} \frac{1}{\gamma_m}) H^i_{kh} + v_{lm} (\dot\partial_j \frac{1}{\gamma_m}) H^i_{kh} \, .$$

Since $\alpha = 0$ and $\beta_l = 0$ reduces an R^h-GR-F_n *to* R^h-recurrent Finsler space and H^i_{kh} of R^h-recurrent Finsler space is *h*-recurrent, so applying these fact in equation (2.25), we have

$$(2.26) \qquad H^i_{jkh|l} - \lambda_l H^i_{jkh} = -[H^r_{kh} \dot\partial_j \Gamma^{*i}_{rl} - H^i_{rh} \dot\partial_j \Gamma^{*r}_{kl} - H^i_{kr} \dot\partial_j \Gamma^{*r}_{hl}$$

$$- H^i_{rkh} P^r_{jl}] + \dot\partial_j \lambda_l H^i_{kh} ,$$

where $\lambda_l = \dfrac{-v_{lm}}{\gamma_m}$.

The above equation shows that

$$H^i_{jkh|l} = \lambda_l H^i_{jkh}$$

if and only if

$$H^r_{kh}\dot{\partial}_j\Gamma^{*i}_{rl} - H^i_{rh}\dot{\partial}_j\Gamma^{*r}_{kl} - H^i_{kr}\dot{\partial}_j\Gamma^{*r}_{hl} - H^i_{rkh}P^r_{jl} - (\dot{\partial}_j\lambda_l)H^i_{kh} = 0 \ . \tag{2.27}$$

Thus, we see that if $\alpha = 0$ and $\beta_l = 0$, theorem (2.3) reduces to

Corollary 2.8. *The Berwald curvature tensor H^i_{jkh} of an R^h-recurrent Finsler space is h-recurrent Finsler space if and only if* (2.27) *holds good.*

Similarly by taking $\gamma_m = 0$ and $\beta_l = 0$ and using the fact that $\gamma_m = 0$ and $\beta_l = 0$ reduces an R^h-GR-F_n to R^h-birecurrent Finsler space and H^i_{kh} of R^h-birecurrent Finsler space is h-birecurrent, the theorem (2.3) reduces to

Corollary 2.9. *The Berwald curvature tensor H^i_{jkh} of an R^h-birecurrent Finsler space is h-birecurrent if and only if*

$$(H^r_{kh}\dot{\partial}_j\Gamma^{*i}_{rm} - H^i_{rh}\dot{\partial}_j\Gamma^{*r}_{km} - H^i_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}$$

$$- H^i_{rkh}P^r_{jm})_{|l} + H^r_{kh|m}\dot{\partial}_j\Gamma^{*i}_{rl} - H^i_{rh|m}\dot{\partial}_j\Gamma^{*r}_{kl} - H^i_{kr|m}\dot{\partial}_j\Gamma^{*r}_{hl}$$

$$- H^i_{kh|r}\dot{\partial}_j\Gamma^{*r}_{ml} - (H^i_{rkh|m} + H^s_{kh}\dot{\partial}_r\Gamma^{*i}_{sm} - H^i_{sh}\dot{\partial}_r\Gamma^{*s}_{km}$$

$$- H^i_{ks}\dot{\partial}_r\Gamma^{*s}_{hm} - H^i_{skh}P^s_{rm})P^r_{jl} = (\dot{\partial}_j a_{lm})H^i_{kh}$$

holds good.

In the similar way taking particular cases (i) $\gamma_m = 0$, (ii) $\beta_l = 0$, (iii) $\nu_{lm} = 0$, $\gamma_m = 0$, and (iv) $\nu_{lm} = 0$, $\beta_l = 0$, theorem (2.3) reduces to the following

Corollary 2.10.

(i) *The Berwald curvature tensor H^i_{jkh} of an R^h-GBR1-F_n is h-GBR1 if and only if*

$$(H^r_{kh}\,\partial_j\,\Gamma^{*i}_{rm} - H^i_{rh}\,\dot{\partial}_j\,\Gamma^{*r}_{km} - H^i_{kr}\,\dot{\partial}_j\,\Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm})_{|l} + H^r_{kh|m}\,\dot{\partial}_j\,\Gamma^{*i}_{rl}$$

$$-H^i_{rh|m}\,\dot{\partial}_j\,\Gamma^{*r}_{kl} - H^i_{kr|m}\,\dot{\partial}_j\,\Gamma^{*r}_{hl} - H^i_{kh|r}\dot{\partial}_j\Gamma^{*r}_{ml} - H^i_{rkh|m}P^r_{jl} - H^s_{kh}\dot{\partial}_r\Gamma^{*i}_{sm}P^r_{jl}$$

$$+H^i_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^r_{jl} + H^i_{ks}\,\dot{\partial}_r\,\Gamma^{*s}_{hm}P^r_{jl} + H^i_{skh}P^s_{rm}P^r_{jl} = (\dot{\partial}_j\,\lambda_l)H^i_{kh|m}$$

$$+\lambda_l H^r_{kh}\,\dot{\partial}_j\,\Gamma^{*i}_{rm} - \lambda_l H^i_{rh}\,\dot{\partial}_j\,\Gamma^{*r}_{km} - \lambda_l H^i_{kr}\,\dot{\partial}_j\,\Gamma^{*r}_{hm} - \lambda_l H^i_{rkh}P^r_{jm} + (\dot{\partial}_j\,a_{lm})H^i_{kh}$$

holds good.

(ii) *The Berwald curvature tensor* H^i_{jkh} *of an* R^h*-GBR2-*F_n *is h-GBR2 if and only if*

$$(H^r_{kh}\,\dot{\partial}_j\,\Gamma^{*i}_{rm} - H^i_{rh}\,\dot{\partial}_j\,\Gamma^{*r}_{km} - H^i_{kr}\,\dot{\partial}_j\,\Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm})_{|l} + H^r_{kh|m}\,\dot{\partial}_j\,\Gamma^{*i}_{rl}$$

$$-H^i_{rh|m}\,\dot{\partial}_j\,\Gamma^{*r}_{kl} - H^i_{kr|m}\,\dot{\partial}_j\,\Gamma^{*r}_{hl} - H^i_{kh|r}\,\dot{\partial}_j\,\Gamma^{*r}_{ml} + H^i_{rkh|m}P^r_{jl}$$

$$-H^s_{kh}\,\dot{\partial}_r\,\Gamma^{*i}_{sm}P^r_{jl} + H^i_{sh}\,\dot{\partial}_r\,\Gamma^{*s}_{km}P^r_{jl} + H^i_{ks}\,\dot{\partial}_r\,\Gamma^{*s}_{hm}P^r_{jl} + H^i_{skh}P^s_{rm}P^r_{jl}$$

$$= (\dot{\partial}_j\,\lambda_m)H^i_{kh|l} + \lambda_m H^r_{kh}\,\dot{\partial}_j\,\Gamma^{*i}_{rl} - \lambda_m H^i_{rh}\,\dot{\partial}_j\,\Gamma^{*r}_{kl} - \lambda_m H^i_{kr}\,\dot{\partial}_j\,\Gamma^{*r}_{hl}$$

$$-\lambda_m H^i_{rkh}P^r_{jl} + (\dot{\partial}_j\,a_{lm})H^i_{kh}$$

holds good.

(iii) *The Berwald curvature tensor* H^i_{jkh} *of an* R^h*-SGBR1-*F_n *is h-SGBR1 if and only if*

$$(H^r_{kh}\,\dot{\partial}_j\,\Gamma^{*i}_{rm} - H^i_{rh}\dot{\partial}_j\Gamma^{*r}_{km} - H^i_{kr}\,\dot{\partial}_j\,\Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm})_{|l} + H^r_{kh|m}\,\dot{\partial}_j\,\Gamma^{*i}_{rl}$$

$$-H^{i}_{rh|lm}\dot{\partial}_j\Gamma^{*r}_{kl}-H^{i}_{kr|lm}\dot{\partial}_j\Gamma^{*r}_{hl}-H^{i}_{kh|lr}\dot{\partial}_j\Gamma^{*r}_{ml}-H^{i}_{rkh|lm}P^{r}_{jl}$$

$$-H^{s}_{kh}\dot{\partial}_r\Gamma^{*i}_{sm}P^{r}_{jl}+H^{i}_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^{r}_{jl}+H^{i}_{ks}\dot{\partial}_r\Gamma^{*s}_{hm}P^{r}_{jl}+H^{i}_{skh}P^{s}_{rm}P^{r}_{jl}$$

$$=(\dot{\partial}_j\lambda_l)H^{i}_{kh|m}+\lambda_l H^{r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-\lambda_l H^{i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-\lambda_l H^{i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-\lambda_l H^{i}_{rkh}P^{r}_{jm}$$

holds good.

(iv) *The Berwald curvature tensor* H^{i}_{jkh} *of an* R^h*-SGBR2-*F_n *is h-SGBR2 if and only if*

$$(H^{r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-H^{i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-H^{i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-H^{i}_{rkh}P^{r}_{jm})_{|l}+H^{r}_{kh|m}\dot{\partial}_j\Gamma^{*i}_{rl}$$

$$-H^{i}_{rh|m}\dot{\partial}_j\Gamma^{*r}_{kl}-H^{i}_{kr|m}\dot{\partial}_j\Gamma^{*r}_{hl}-H^{i}_{kh|r}\dot{\partial}_j\Gamma^{*r}_{ml}-H^{i}_{rkh|m}P^{r}_{jl}$$

$$-H^{s}_{kh}\dot{\partial}_r\Gamma^{*i}_{sm}P^{r}_{jl}+H^{i}_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^{r}_{jl}+H^{i}_{ks}\dot{\partial}_r\Gamma^{*s}_{hm}P^{r}_{jl}$$

$$+H^{i}_{skh}P^{s}_{rm}P^{r}_{jl}=(\dot{\partial}_j\lambda_m)H^{i}_{kh|l}+\lambda_m H^{r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rl}-\lambda_m H^{i}_{rh}\dot{\partial}_j\Gamma^{*r}_{kl}$$

$$-\lambda_m H^{i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hl}-\lambda_m H^{i}_{rkh}P^{r}_{jl}$$

holds good.

Differentiating (1.8.8b) covariantly with respect to x^m in the sense of Cartan, we have

$$R^{i}_{jkh|m}=K^{i}_{jkh|m}+C^{i}_{jr|m}H^{r}_{kh}+C^{i}_{jr}H^{r}_{kh|m}\,. \tag{2.28}$$

Again differentiating (2.28) covariantly with respect to x^l in the sense of Cartan and using (2.4) we get

$$(2.29) \qquad -\beta_l R^i_{jkh|m} - \gamma_m R^i_{jkh|l} - v_{lm} R^i_{jkh} = \alpha K^i_{jkh|m|l} + \alpha C^i_{jr|m|l} H^r_{kh}$$

$$+\alpha C^i_{jr|m} H^r_{kh|l} + \alpha C^i_{jr|l} H^r_{kh|m} + \alpha C^i_{jr} H^r_{kh|m|l} \;.$$

Since the tensor H^r_{kh} of an R^h-GR-F_n is h-GR, (2.29) implies

$$-\beta_l (R^i_{jkh|m} - C^i_{jr} H^r_{kh|m}) - \gamma_m (R^i_{jkh|l} - C^i_{jr} H^i_{kh|l})$$

$$-v_{lm} (R^i_{jkh} - C^i_{jr} H^r_{kh}) = \alpha K^i_{jkh|m|l} + \alpha C^i_{jr|m|l} H^r_{kh}$$

$$+\alpha C^i_{jr|m} H^r_{kh|l} + \alpha H^r_{kh|m} C^i_{jr|l}$$

which may be rewritten as

$$-\beta_l (R^i_{jkh|m} - C^i_{jr} H^r_{kh|m} - C^i_{jr|m} H^r_{kh}) - \gamma_m (R^i_{jkh|l}$$

$$-C^i_{jr} H^r_{kh|l} - C^i_{jr|l} H^r_{kh}) - v_{lm} (R^i_{jkh} - C^i_{jr} H^r_{kh})$$

$$= \alpha K^i_{jkh|m|l} + \alpha C^i_{jr|m|l} H^r_{kh} + \alpha C^i_{jr|m} H^r_{kh|l} + \alpha H^r_{kh|m} C^i_{jr|l}$$

$$+\beta_l C^i_{jr|m} H^r_{kh} + \gamma_m H^r_{kh} C^i_{jr|l} \;.$$

This equation, in view of (1.8.8b) and (2.28), gives

$$(2.30) \qquad \alpha K^i_{jkh|m|l} + \beta_l K^i_{jkh|m} + \gamma_m K^i_{jkh|l} + v_{lm} K^i_{jkh} = -[(\alpha C^i_{jr|m|l}$$

$$+\beta_l C^i_{jr|m} + \gamma_m C^i_{jr|l}) H^r_{kh} + \alpha C^i_{jr|m} H^r_{kh|l} + \alpha H^r_{kh|m} C^i_{jr|l}] \,.$$

This shows that

$$(2.31) \qquad \alpha K^i_{jkh|m|l} + \beta_l K^i_{jkh|m} + \gamma_m K^i_{jkh|l} + v_{lm} K^i_{jkh} = 0$$

if and only if

$$(\alpha C^{i}_{jr|m|l} + \beta_l C^{i}_{jr|m} + \gamma_m C^{i}_{jr|l}) H^{r}_{kh} + \alpha (C^{i}_{jr|m} H^{r}_{kh|l} + H^{r}_{kh|m} C^{i}_{jr|l}) = 0 . \tag{2.32}$$

Thus, we have

Theorem 2.4. *Cartan curvature tensor* K^{i}_{jkh} *of an* R^h*-GR-*F_n *is h-GR if and only if the condition* (2.32) *is satisfied.*

If we take $\alpha = 0$ and $\beta_l = 0$ in (2.30), we have

$$\gamma_m K^{i}_{jkh|l} + \nu_{lm} K^{i}_{jkh} = -\gamma_m C^{i}_{jr|l} H^{r}_{kh} \tag{2.33}$$

which may be written as

$$K^{i}_{jkh|l} = \lambda_l K^{i}_{jkh} - C^{i}_{jr|l} H^{r}_{kh} . \tag{2.34}$$

This shows that

$$K^{i}_{jkh|l} = \lambda_l K^{i}_{jkh}$$

if and only if

$$C^{i}_{jr|l} H^{r}_{kh} = 0 .$$

Thus, we see that $\alpha = 0$ and $\beta_l = 0$ reduces theorem 2.4 to

Corollary 2.11. *Cartan curvature tensor* K^{i}_{jkh} *of an* R^h*-recurrent space is h-recurrent if and only if*

$$C^{i}_{jr|l} H^{r}_{kh} = 0 .$$

Similarly considering the cases (i) $\gamma_m = 0, \beta_l = 0$ (ii) $\gamma_m = 0$, (iii) $\beta_l = 0$, (iv) $\nu_{lm} = 0$, $\gamma_m = 0$ and (v) $\nu_{lm} = 0, \beta_l = 0$, we find that the theorem 2.4 reduces to

Corollary 2.12.

(i) *Cartan curvature tensor* K^i_{jkh} *of an* R^h*-birecurrent space is h-birecurrent if and only if*

$$H^r_{kh}C^i_{jr|m|l}+C^i_{jr|m}H^r_{kh|l}+C^i_{jr|l}H^r_{kh|m}=0$$

holds good.

(ii) *Cartan curvature tensor* K^i_{jkh} *of an* R^h*-GBR1-* F_n *is h-GBR1 if and only if*

$$C^i_{jr|m|l}H^r_{kh}-\lambda_l C^i_{jr|m}H^r_{kh}+C^i_{jr|m}H^r_{kh|l}+C^i_{jr|l}H^r_{kh|m}=0$$

holds good.

(iii) *Cartan curvature tensor* K^i_{jkh} *of an* R^h*-GBR2-* F_n *is h-GBR2 if and only if*

$$C^i_{jr|m|l}H^r_{kh}-\lambda_m C^i_{jr|l}H^r_{kh}+C^i_{jr|l}H^r_{kh|m}+C^i_{jr|m}H^r_{kh|l}=0$$

holds good.

(iv) *Cartan curvature tensor* K^i_{jkh} *of an* R^h*-SGBR1-* F_n *is h-SGBR1 if and only if*

$$(C^i_{jr|m|l}-\lambda_l C^i_{jr|m})H^r_{kh}+C^i_{jr|m}H^r_{kh|l}+C^i_{jr|l}H^r_{kh|m}=0$$

holds good.

(v) *Cartan curvature tensor* K^i_{jkh} *of an* R^h*-SGBR2-* F_n *is h-SGBR2 if and only if*

$$(C^i_{jr|m|l}-\lambda_m C^i_{jr|l})H^r_{kh}+C^i_{jr|m}H^r_{kh|l}+C^i_{jr|l}H^r_{kh|m}=0$$

holds good.

3. Certain Identities

We know that the tensor R^i_{jkh} satisfies the identity [14, 15]

(3.1) $$R_{ijkh} + R_{ihkj} + R_{ikjh} + (C_{ijs} K^{s}_{rhk} + C_{ihs} K^{s}_{rkj} + C_{iks} K^{s}_{rjh}) y^{r} = 0 \,.$$

Using equation (1.8.8c) in (3.1), we get

(3.2) $$R_{ijkh} + R_{ihkj} + R_{ikjh} + (C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh}) = 0 \,.$$

Differentiating (3.2) covariantly with respect to x^{m} in the sense of Cartan, we get

(3.3) $$R_{ijkh|m} + R_{ihkj|m} + R_{ikjh|m} + (C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh})_{|m} = 0.$$

Differentiating (3.3) covariantly with respect to x^{l} in the sense of Cartan, we get

(3.4) $$R_{ijkh|m|l} + R_{ihkj|m|l} + R_{ikjh|m|l} + (C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh})_{|m|l} = 0 \,.$$

Multiplying (3.4) by α and using (2.5), we have

(3.5) $$\begin{aligned} &-\beta_{l}(R_{ijkh|m} + R_{ihkj|m} + R_{ikjh|m}) - \gamma_{m}(R_{ijkh|l} \\ &+ R_{ihkj|l} + R_{ikjh|l}) - \nu_{lm}(R_{ijkh} + R_{ihkj} + R_{ikjh}) \\ &+ \alpha(C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh})_{|m|l} = 0 \,. \end{aligned}$$

Using (3.2) and (3.3) in (3.5), we get

(3.6) $$\begin{aligned} &\beta_{l}(C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh})_{|m} + \gamma_{m} \\ &(C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh})_{|l} + \nu_{lm}(C_{ijs} H^{s}_{hk} \\ &+ C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh}) + \alpha(C_{ijs} H^{s}_{hk} + C_{ihs} H^{s}_{kj} + C_{iks} H^{s}_{jh})_{|m|l} = 0 \,. \end{aligned}$$

Transvecting (3.6) by y^{j} and using (1.5.2a) and (1.9.7b), we get

$$(3.7)\qquad \alpha(C_{iks}H^s_h - C_{ihs}H^s_k)_{|m|l} + \beta_l(C_{iks}H^s_h - C_{ihs}H^s_k)_{|m}$$

$$+\gamma_m(C_{iks}H^s_h - C_{ihs}H^s_k)_{|l} + \nu_{lm}(C_{iks}H^s_h - C_{ihs}H^s_k) = 0.$$

Multiplying (3.7) by g^{pi} and using (1.5.3) and the symmetric property of *(h)*-*hv*-torsion tensor C_{ijk} in all its lower indices, we have

$$(3.8)\qquad \alpha(C^p_{ks}H^s_h - C^p_{hs}H^s_k)_{|m|l} + \beta_l(C^p_{ks}H^s_h - C^p_{hs}H^s_k)_{|m}$$

$$+\gamma_m(C^p_{ks}H^s_h - C^p_{hs}H^s_k)_{|l} + \nu_{lm}(C^p_{ks}H^s_h - C^p_{hs}H^s_k) = 0.$$

In view of (1.5.3) and (1.8.9a), the equation (3.3) may be written as

$$(3.9)\qquad R^i_{jkh|m} + R^i_{hkj|m} + R^i_{kjh|m} + (C^i_{js}H^s_{hk} + C^i_{hs}H^s_{kj} + C^i_{ks}H^s_{jh})_{|m} = 0.$$

Using (1.8.8b) and (1.8.8c) in (3.9), we get

$$(3.10)\qquad K^i_{jhk|m} + K^i_{hkj|m} + K^i_{kjh|m} + 2(C^i_{js}H^s_{hk} + C^i_{hs}H^s_{kj} + C^i_{ks}H^s_{jh})_{|m} = 0.$$

In view of (1.8.3a) the equation (3.10) takes the form

$$(3.11)\qquad C^i_{js|m}H^s_{hk} + C^i_{js}H^s_{hk|m} + C^i_{hs|m}H^s_{kj} + C^i_{hs}H^s_{kj|m} + C^i_{ks|m}H^s_{jh} + C^i_{ks}H^s_{jh|m} = 0.$$

Differentiating (3.11) covariantly with respect to x^l in the sense of Cartan, we get

$$(3.12)\qquad C^i_{js|m|l}H^s_{hk} + C^i_{js|m}H^s_{hk|l} + C^i_{js|l}H^s_{hk|m} + C^i_{js}H^s_{hk|m|l}$$

$$+ C^i_{ks|m|l}H^s_{jh} + C^i_{ks|l}H^s_{jh|m} + C^i_{ks|m}H^s_{jh|l} + C^i_{ks}H^s_{jh|m|l}$$

$$+ C^i_{hs|l}H^s_{kj|m} + C^i_{hs}H^s_{kj|m|l} + C^i_{hs|m}H^s_{kj|l} + C^i_{hs|m|l}H^s_{kj} = 0.$$

Multiplying (3.12) by y^m and using (1.8.10), we get

$$(3.13)\qquad P^i_{js|l}H^s_{hk} + P^i_{js}H^s_{hk|l} + C^i_{js|l}H^s_{hk|m}y^m + C^i_{js}H^s_{hk|m|l}y^m$$

$$+ P^i_{hs|l}H^s_{kj} + P^i_{hs}H^s_{kj|l} + C^i_{hs|l}H^s_{kj|m}y^m + C^i_{hs}H^s_{kj|m|l}y^m$$

$$+ P^i_{ks|l}H^s_{jh} + P^i_{ks}H^s_{jh|l} + C^i_{ks|l}H^s_{jh|m}y^m + C^i_{ks}H^s_{jh|m|l}y^m = 0.$$

Multiplying (3.13) by y^h and using (1.5.2c), (1.9.7b) and $P^i_{kh}y^h = 0$, we have

$$(3.14)\qquad (P^i_{js}H^s_k)_{|l} - (P^i_{ks}H^s_j)_{|l} + (C^i_{js}H^s_{k|m} - C^i_{ks}H^s_{j|m})_{|l}\, y^m = 0.$$

Differentiating (1.8.8a) covariantly with respect to x^l in the sense of Cartan, we get

$$(3.15)\qquad R^i_{jkh|m|l} + R^i_{jmk|h|l} + R^i_{jhm|k|l} + y^r(R^s_{rhm}P^i_{jks} + R^s_{rkh}P^i_{jms} + R^s_{rmk}P^i_{jhs})_{|l} = 0\ .$$

In view of (2.4) and (1.8.8c), (3.15) may be written as

$$(3.16)\qquad -\beta_l R^i_{jkh|m} - \gamma_m R^i_{jkh|l} - v_{lm}R^i_{jkh} - \beta_l R^i_{jmk|h} - \gamma_h R^i_{jmk|l}$$

$$-v_{lh}R^i_{jmk} - \beta_l R^i_{jhm|k} - \gamma_k R^i_{jhm|l} - v_{lk}R^i_{jhm}$$

$$+\alpha(H^s_{hm}P^i_{jks} + H^s_{kh}P^i_{jms} + H^s_{mk}P^i_{jhs})_{|l} = 0.$$

Transvecting (3.16) by y^j and using (1.8.8c) and (1.8.10), we get

$$-\beta_l H^i_{kh|m} - \gamma_m H^i_{kh|l} - v_{lm}H^i_{kh} - \beta_l H^i_{mk|h} - \gamma_h H^i_{mk|l}$$

$$-v_{lh}H^i_{mk} - \beta_l H^i_{hm|k} - \gamma_k H^i_{hm|l} - v_{lk}H^i_{hm} + \alpha(H^s_{hm}P^i_{ks}$$

$$+H^s_{kh}P^i_{ms} + H^s_{mk}P^i_{hs})_{|l} = 0,$$

which may be rewritten as

(3.17) $$-\beta_l(H^i_{khlm}+H^i_{mklh}+H^i_{hmlk})-(v_{lm}H^i_{kh}+v_{lh}H^i_{mk}$$

$$+v_{lk}H^i_{hm})-\gamma_m H^i_{khll}-\gamma_h H^i_{mkll}-\gamma_k H^i_{hmll}+\alpha(H^s_{hm}P^i_{ks}$$

$$+H^s_{kh}P^i_{ms}+H^s_{mk}P^i_{hs})_{|l}=0\ .$$

Thus, we conclude

Theorem 3.1. *In an* R^h*-GR-*F_n *the identities* (3.13), (3.14) *and* (3.17) *hold and the tensors* $C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh}, C_{iks}H^s_h-C_{ihs}H^s_k$ and $C^p_{ks}H^s_h-C^p_{hs}H^s_k$ are *h-GR.*

Putting $\alpha=0$ and $\beta_l=0$ in (3.6) we find

$$\gamma_m(C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh})_{|l}+v_{lm}(C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh})=0\,,$$

which may be written as

(3.18) $$(C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh})_{|l}=\lambda_l(C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh})\,,$$

where $\lambda_l=\dfrac{-v_{lm}}{\gamma_m}$.

Also putting $\alpha=0$ and $\beta_l=0$ in equation (3.7) and (3.8), we get

$$(C_{iks}H^s_h-C_{ihs}H^s_k)_{|l}=\lambda_l(C_{iks}H^s_h-C_{ihs}H^s_k)$$

and

$$(C^p_{ks}H^s_h-C^p_{hs}H^s_k)_{|l}=\lambda_l(C^p_{ks}H^s_h-C^p_{hs}H^s_k)$$

respectively.

Thus $\alpha=0$ and $\beta_l=0$ reduces theorem 3.1 to

Corollary 3.1. *In an* R^h*-recurrent space the tensors* $C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh}$, $C_{iks}H^s_h-C_{ihs}H^s_k$ *and* $C^p_{ks}H^s_h-C^p_{hs}H^s_k$ *are all h-recurrent.*

Similarly by taking (i) $\gamma^{'s}_{m=0}$ and $\beta_l=0$, (ii) $\gamma^{'s}_m=0$, (iii) $\beta_l=0$ (iv) $v^{'s}_{lm}=0$ and $\gamma^{'s}_m=0$ and (v) $v^{'s}_{lm}=0$ and $\beta_l=0$, theorem 3.1 reduces to

Corollary 3.2.

(i) *In an* R^h*-birecurrent space the identity*

$$a_{lm}H^i_{kh}+a_{lh}H^i_{mk}+a_{lk}H^i_{hm}+(H^s_{hm}P^i_{ks}+H^s_{kh}P^i_{ms}+H^s_{mk}P^i_{hs})_{|l}=0$$

holds and the tensors $C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh}$, $C_{iks}H^s_h-C_{ihs}H^s_k$ *and* $C^p_{ks}H^s_h-C^p_{hs}H^s_k$ *are h-birecurrent.*

(ii) *In an* $R^h-GBR1-F_n$ *the identity*

$$\lambda_l(H^i_{kh|m}+H^i_{mk|h}+H^i_{hm|k})+(a_{lm}H^i_{kh}+a_{lh}H^i_{mk}+a_{lk}H^i_{hm})$$

$$+(H^s_{hm}P^i_{ks}+H^s_{kh}P^i_{ms}+H^s_{mk}P^i_{hs})_{|l}=0$$

holds and the tensors $C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh}$, $C_{iks}H^s_h-C_{ihs}H^s_k$ *and* $C^p_{ks}H^s_h-C^p_{hs}H^s_k$ *are h-GBR1.*

(iii) *In an* $R^h-GBR2-F_n$ *the identity*

$$(\lambda_m H^i_{kh|l}+\lambda_h H^i_{mk|l}+\lambda_k H^i_{hm|l})+(a_{lm}H^i_{kh}+a_{lh}H^i_{mk}$$

$$+a_{lk}H^i_{hm})+(H^s_{hm}P^i_{ks}+H^s_{kh}P^i_{ms}+H^s_{mk}P^i_{hs})_{|l}=0$$

holds and the tensors $C_{ijs}H^s_{hk}+C_{ihs}H^s_{kj}+C_{iks}H^s_{jh}$, $C_{iks}H^s_h-C_{ihs}H^s_k$ *and* $C^p_{ks}H^s_h-C^p_{hs}H^s_k$ *are h-GBR2.*

(iv) *In an* R^h*-SGBR1-*F_n *the identity*

$$\lambda_l (H^i_{khlm} + H^i_{mklh} + H^i_{hmlk}) + (H^s_{hm} P^i_{ks} + H^s_{kh} P^i_{ms} + H^s_{mk} P^i_{hs})_{|l} = 0$$

holds and the tensors $C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh}$, $C_{iks} H^s_h - C_{ihs} H^s_k$ *and* $C^p_{ks} H^s_h - C^p_{hs} H^s_k$ *are h-SGBR1.*

(v) *In an* R^h*-SGBR2-*F_n *the identity*

$$(\lambda_m H^i_{khl|l} + \lambda_h H^i_{mkl|l} + \lambda_k H^i_{hml|l}) + (H^s_{hm} P^i_{ks} + H^s_{kh} P^i_{ms} + H^s_{mk} P^i_{hs})_{|l} = 0$$

holds and the tensors $C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh}$, $C_{iks} H^s_h - C_{ihs} H^s_k$ *and* $C^p_{ks} H^s_h - C^p_{hs} H^s_k$ *are h-SGBR2.*

4. P2-Like Generalized *h*-Recurrent Spaces

A P2-like Finsler space is characterized by

(4.1) $$P^i_{jkh} = \phi_j C^i_{kh} - \phi^i C_{jkh}.$$

where ϕ_j is a non-zero covariant vector field. A P2-like space is necessarily a P^*-Finsler space which is characterized by

(4.2) $$P^i_{kh} = \phi C^i_{kh},$$

where $P^i_{kh} = C^i_{kh|s} y^s$.

Consider a P2-like R^h-GR-F_n. In this space we have the equations (4.1), (4.2) and the identity (1.8.8a).

Putting (4.1) in (1.8.8a), we get

(4.3) $$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + y^r R^s_{rhm} (\phi_j C^i_{ks} - \phi^i C_{jks})$$

$$+\, y^r R^s_{rkh}(\phi_j C^i_{ms} - \phi^i C_{jms}) + y^r R^s_{rmk}(\phi_j C^i_{hs} - \phi^i C_{jhs}) = 0 .$$

Using (1.8.8c) in (4.3), we have

$$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \phi_j (H^s_{hm} C^i_{ks} + H^s_{kh} C^i_{ms} \tag{4.4}$$

$$+ H^s_{mk} C^i_{hs}) - \phi^i (H^s_{hm} C_{jks} + H^s_{kh} C_{jms} + H^s_{mk} C_{jhs}) = 0 .$$

In view of (1.8.8b), (1.8.3a) and (1.8.9b), the equation (4.4) turns into

$$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \phi_j (R^i_{hmk} + R^i_{khm} + R^i_{mkh})$$

$$-\phi^i (R_{jmkh} + R_{jhmk} + R_{jkhm}) = 0 ,$$

which implies

$$R_{ijkh|m} + R_{ijmk|h} + R_{ijhm|k} = \phi_i (R_{jmkh} + R_{jhmk} + R_{jkhm}) \tag{4.5}$$

$$- \phi_j (R_{ihmk} + R_{ikhm} + R_{imkh}) .$$

Transvecting (4.5) by y^i, we get

$$H_{\cdot jkh|m} + H_{\cdot jmk|h} + H_{\cdot jhm|k} = \phi (R_{jmkh} + R_{jhmk} + R_{jkhm}) \tag{4.6}$$

$$- \phi_j (H_{\cdot hmk} + H_{\cdot khm} + H_{\cdot mkh}) ,$$

where $\phi = \phi_i y^i$.

Now differentiating (1.9.12a) partially with respect to y^j and using (1.9.3), we get

$$g_{ji} H^i_{kh} + y_i H^i_{jkh} = 0 . \tag{4.7}$$

Taking skew-symmetric part of (4.7) with respect to the indices *j*, *k*, *h* and using (1.9.9a), we get

$$+ y^r R^s_{rkh}(\phi_j C^i_{ms} - \phi^i C_{jms}) + y^r R^s_{rmk}(\phi_j C^i_{hs} - \phi^i C_{jhs}) = 0.$$

Using (1.8.8c) in (4.3), we have

(4.4) $$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \phi_j (H^s_{hm} C^i_{ks} + H^s_{kh} C^i_{ms}$$

$$+ H^s_{mk} C^i_{hs}) - \phi^i (H^s_{hm} C_{jks} + H^s_{kh} C_{jms} + H^s_{mk} C_{jhs}) = 0.$$

In view of (1.8.8b), (1.8.3a) and (1.8.9b), the equation (4.4) turns into

$$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \phi_j (R^i_{hmk} + R^i_{khm} + R^i_{mkh})$$

$$-\phi^i (R_{jmkh} + R_{jhmk} + R_{jkhm}) = 0,$$

which implies

(4.5) $$R_{ijkh|m} + R_{ijmk|h} + R_{ijhm|k} = \phi_i (R_{jmkh} + R_{jhmk} + R_{jkhm})$$

$$- \phi_j (R_{ihmk} + R_{ikhm} + R_{imkh}).$$

Transvecting (4.5) by y^i, we get

(4.6) $$H_{\cdot jkh|m} + H_{\cdot jmk|h} + H_{\cdot jhm|k} = \phi (R_{jmkh} + R_{jhmk} + R_{jkhm})$$

$$- \phi_j (H_{\cdot hmk} + H_{\cdot khm} + H_{\cdot mkh}),$$

where $\phi = \phi_i y^i$.

Now differentiating (1.9.12a) partially with respect to y^j and using (1.9.3), we get

(4.7) $$g_{ji} H^i_{kh} + y_i H^i_{jkh} = 0.$$

Taking skew-symmetric part of (4.7) with respect to the indices *j*, *k*, *h* and using (1.9.9a), we get

(4.8) $$g_{ij}H^i_{kh} + g_{ih}H^i_{jk} + g_{ik}H^i_{hj} = 0\,.$$

Using (1.9.13) in (4.8), we may write

(4.9) $$H_{\cdot jkh} + H_{\cdot hjk} + H_{\cdot khj} = 0\,.$$

Using (4.9) in (4.6), we get

$$H_{\cdot jkh|m} + H_{\cdot jmk|h} + H_{\cdot jhm|k} = \phi(R_{jmkh} + R_{jhmk} + R_{jkhm})$$

from which we find

(4.10) $$H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} = \phi(R^i_{mkh} + R^i_{hmk} + R^i_{khm})\,.$$

Differentiating (4.10) covariantly with respect to x^l in the sense of Cartan and using (2.14), we get

(4.11) $$-\beta_l(H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k}) - \gamma_m H^i_{kh|l} - \gamma_h H^i_{mk|l}$$

$$-\gamma_k H^i_{hm|l} - v_{lm}H^i_{kh} - v_{lh}H^i_{mk} - v_{lk}H^i_{hm} = \alpha\phi_{|l}(R^i_{mkh}$$

$$+ R^i_{hmk} + R^i_{khm}) + \alpha\phi(R^i_{mkh} + R^i_{hmk} + R^i_{khm})_{|l}\,.$$

Thus, we may conclude

Theorem 4.1. *The identities* (4.10) *and* (4.11) *are satisfied in a P2-like*

If we take $\gamma_m^{'s} = 0$ and $\beta_l = 0$ in (4.11), we have

(4.12) $$a_{lm}H^i_{kh} + a_{lh}H^i_{mk} + a_{lk}H^i_{hm} = \phi_{|l}(R^i_{mkh} + R^i_{hmk} + R^i_{khm})$$

$$+\phi(R^i_{mkh} + R^i_{hmk} + R^i_{khm})_{|l}\,,$$

where $a_{lm} = \dfrac{-v_{lm}}{\alpha}$.

Thus $\beta_l = 0$ and $\gamma_m^{'s} = 0$ reduce theorem 4.1 to

Corollary 4.1. *The identity* (4.12) *is satisfied in a P2-like R^h-birecurrent space.*

Similarly taking (i) $\gamma_m^{'s} = 0$, (ii) $\beta_l = 0$, (iii) $v_{lm}^{'s} = 0$, $\gamma_m^{'s} = 0$ and (iv) $v_{lm}^{'s} = 0, \beta_l = 0$, we find that theorem 4.1 reduces to

Corollary 4.2.

(i) *In a P2-like R^h-GBR1-F_n we have the identity*

$$\lambda_l (H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k}) + a_{lm} H^i_{kh} + a_{lh} H^i_{mk}$$

$$+ a_{lk} H^i_{hm} = \phi_{|l} (R^i_{mkh} + R^i_{hmk} + R^i_{khm}) + \phi (R^i_{mkh} + R^i_{hmk} + R^i_{khm})_{|l},$$

where $\lambda_l = \dfrac{-\beta_l}{\alpha}$ *and* $a_{lm} = \dfrac{-v_{lm}}{\alpha}$.

(ii) *In a P2-like R^h-GBR2-F_n we have the identity*

$$\lambda_m H^i_{kh|l} + \lambda_h H^i_{mk|l} + \lambda_k H^i_{hm|l} + a_{lm} H^i_{kh} + a_{lh} H^i_{mk}$$

$$+ a_{lk} H^i_{hm} = \phi_{|l} (R^i_{mkh} + R^i_{hmk} + R^i_{khm}) + \phi (R^i_{mkh} + R^i_{hmk} + R^i_{khm})_{|l},$$

where $\lambda_m = \dfrac{-\gamma_m}{\alpha}$ *and* $a_{lm} = \dfrac{-v_{lm}}{\alpha}$.

(iii) *In a P2-like R^h-SGBR1-F_n we have the identity*

$$\lambda_l (H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k}) = \phi_{|l} (R^i_{mkh} + R^i_{hmk} + R^i_{khm})$$

$$+ \phi (R^i_{mkh} + R^i_{hmk} + R^i_{khm})_{|l} .$$

(iv) *In a P2-like* R^h*-SGBR2-*F_n *we have the identity*

$$\lambda_m H^i_{khl|l} + \lambda_h H^i_{mkl|l} + \lambda_k H^i_{hml|l} = \phi_{|l}(R^i_{mkh} + R^i_{hmk} + R^i_{khm})$$

$$+\phi(R^i_{mkh} + R^i_{hmk} + R^i_{khm})_{|l}.$$

5. Some Theorems Regarding Projection of Curvature Tensor R^i_{jkh} on Indicatrix

The projection of any tensor T^i_j on the indicatrix is given by

(5.1) a) $$p.T^i_j = T^\alpha_\beta h^i_\alpha h^\beta_j ,$$

where

b) $$h^i_\alpha := \delta^i_\alpha - l^i l_\alpha .$$

If the projection of a tensor T^i_j on the indicatrix I_{n-1} is the same tensor T^i_j, the tensor is called an *indicatric tensor*. The tensors $H^i_k, h^i_k, G^i_{jk}, P^i_{jk}$ and S^i_{jkh} are example of indicatric tensors. The projection of the vector y^i, l^i and the metric tensor g_{ij} on the indicatrix are given by

(5.2) a) $$p.y^i = 0,$$

b) $$p.l^i = 0$$

and

c) $$p.g_{ij} = h_{ij},$$

where

d) $$h_{ij} := g_{ij} - l_i l_j .$$

Let us consider a Finsler space F_n for which the Cartan third curvature tensor R^i_{jkh} is generalized recurrent which is characterized by (2.4). In view of (5.1a), the projection of the tensor R^i_{jkh} on the indicatrix is given by

(5.3) $$p.R^i_{jkh} = R^a_{bcd} h^i_a h^b_j h^c_k h^d_h .$$

Taking h-covariant derivative of (5.3) twice with respect to x^m and x^l successively and using the fact that $h^i_{a|m} = 0$, we get

(5.4) $$(p.R^i_{jkh})_{|m|l} = R^a_{bcd|m|l} h^i_a h^b_j h^c_k h^d_h .$$

Transvecting (5.4) by α and using (2.4), we have

(5.5) $$\alpha(p.R^i_{jkh})_{|m|l} = (-\beta_l R^a_{bcd|m} - \gamma_m R^a_{bcd|l} - \nu_{lm} R^a_{bcd}) h^i_a h^b_j h^c_k h^d_h .$$

Using the definition of projection of indicatrix in (5.5) and the fact that $h^i_{a|m} = 0$, we get

(5.6) $$\alpha(p.R^i_{jkh})_{|m|l} + \beta_l (p.R^i_{jkh})_{|m} + \gamma_m (p.R^i_{jkh})_{|l} + \nu_{lm} (p.R^i_{jkh}) = 0 .$$

This shows that $p.R^i_{jkh}$ is generalized recurrent. Thus we conclude

Theorem 5.1. *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-GR-F_n on indicatrix is generalized recurrent.*

Let us take α = 0 and $\beta_l = 0$ in (5.6), then we have

$$\gamma_m (p.R^i_{jkh})_{|l} = -\nu_{lm} (p.R^i_{jkh}) ,$$

which may be written as

(5.7) $$(p.R^i_{jkh})_{|l} = \lambda_l (p.R^i_{jkh}).$$

Thus $\alpha = 0$ and $\beta_l = 0$ reduce theorem 5.1 to

Corollary 5.1. *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-recurrent space on indicatrix is recurrent.*

Similarly taking (i) $\beta_l = 0$ and $\gamma_m = 0$, (ii) $\gamma_m = 0$, (iii) $\beta_l = 0$, (iv) $\nu_{lm} = 0$ and $\gamma_m = 0$ and (v) $\nu_{lm} = 0$ and $\beta_l = 0$ in (5.6), theorem 5.1 reduces to

Corollary 5.2.

(i) *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-birecurrent space on indicatrix is birecurrent.*

(ii) *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-GBR1-F_n on indicatrix is GBR1.*

(iii) *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-GBR2-F_n on indicatrix is GBR2.*

(iv) *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-SGBR1-F_n on indicatrix is SGBR1.*

(v) *The projection of the Cartan third curvature tensor R^i_{jkh} of an R^h-SGBR2-F_n on indicatrix is SGBR2.*

Let us consider a Finsler space F_n for which the associate tensor R_{jrkh} of the Cartan third curvature tensor R^i_{jkh} is *h-GR* i.e. characterized by (2.5). In view of (5.1a), the projection of the tensor R_{jrkh} on the indicatrix is given by

(5.8) $$p.R_{jrkh} = R_{abcd} h^a_j h^b_r h^c_k h^d_h .$$

Taking *h*-covariant derivative of (5.8) twice with respect to x^m and x^l successively and using the fact that $h^i_{a|m} = 0$, we get

$$(5.9)\qquad (p.R_{jrkh})_{|m|l} = R_{abcd|m|l} h_j^a h_r^b h_k^c h_h^d .$$

Transvecting (5.9) by α and using (2.5), we get

$$(5.10)\qquad \alpha(p.R_{jrkh})_{|m|l} + (\beta_l R_{abcd|m} + \gamma_m R_{abcd|l} + \nu_{lm} R_{abcd}) h_j^a h_r^b h_k^c h_h^d .$$

Using the definition of projection on indicatrix in (5.10) and the fact that $h^i_{a|m} = 0$, we have

$$(5.11)\qquad \alpha(p.R_{jrkh})_{|m|l} + \beta_l (p.R_{jrkh})_{|m} + \gamma_m (p.R_{jrkh})_{|l} + \nu_{lm} (p.R_{jrkh}) = 0 .$$

This shows that $p.R_{jrkh}$ is generalized recurrent. Thus, we conclude

Theorem 5.2. *The projection of the tensor R_{jrkh} (the associate tensor of the Cartan curvature tensor R^i_{jkh}) of an R^h-GR-F_n on indicatrix is generalized recurrent.*

Let us consider a Finsler space F_n for which the $h(v)$-torsion tensor H^i_{kh} is *h-GR*. Obviously this space is characterized by (2.14). In view of (5.1a), the projection of the $h(v)$-torsion tensor H^i_{kh} on the indicatrix is given by

$$(5.12)\qquad p.H^i_{kh} = H^a_{bc} h^i_a h^b_k h^c_h .$$

Taking h-covariant derivative of (5.12) twice with respect to x^m and x^l successively and using the fact that $h^i_{a|m} = 0$, we get

$$(5.13)\qquad (p.H^i_{kh})_{|m|l} = H^a_{bc|m|l} h^i_a h^b_k h^c_h .$$

Transvecting (5.13) by α and using (2.14), we get

$$(5.14)\qquad \alpha(p.H^i_{kh})_{|m|l} = -(\beta_l H^a_{bc|m} + \gamma_m H^a_{bc|l} + \nu_{lm} H^a_{bc}) h^i_a h^b_k h^c_h .$$

(5.9) $$(p.R_{jrkh})_{|m|l} = R_{abcd|m|l} h_j^a h_r^b h_k^c h_h^d .$$

Transvecting (5.9) by α and using (2.5), we get

(5.10) $$\alpha(p.R_{jrkh})_{|m|l} + (\beta_l R_{abcd|m} + \gamma_m R_{abcd|l} + \nu_{lm} R_{abcd}) h_j^a h_r^b h_k^c h_h^d .$$

Using the definition of projection on indicatrix in (5.10) and the fact that $h^i_{a|m} = 0$, we have

(5.11) $$\alpha(p.R_{jrkh})_{|m|l} + \beta_l (p.R_{jrkh})_{|m} + \gamma_m (p.R_{jrkh})_{|l} + \nu_{lm}(p.R_{jrkh}) = 0 .$$

This shows that $p.R_{jrkh}$ is generalized recurrent. Thus, we conclude

Theorem 5.2. *The projection of the tensor R_{jrkh} (the associate tensor of the Cartan curvature tensor R^i_{jkh}) of an R^h-GR-F_n on indicatrix is generalized recurrent.*

Let us consider a Finsler space F_n for which the $h(v)$-torsion tensor H^i_{kh} is *h-GR*. Obviously this space is characterized by (2.14). In view of (5.1a), the projection of the $h(v)$-torsion tensor H^i_{kh} on the indicatrix is given by

(5.12) $$p.H^i_{kh} = H^a_{bc} h^i_a h^b_k h^c_h .$$

Taking h-covariant derivative of (5.12) twice with respect to x^m and x^l successively and using the fact that $h^i_{a|m} = 0$, we get

(5.13) $$(p.H^i_{kh})_{|m|l} = H^a_{bc|m|l} h^i_a h^b_k h^c_h .$$

Transvecting (5.13) by α and using (2.14), we get

(5.14) $$\alpha(p.H^i_{kh})_{|m|l} = -(\beta_l H^a_{bc|m} + \gamma_m H^a_{bc|l} + \nu_{lm} H^a_{bc}) h^i_a h^b_k h^c_h .$$

Using the definition of projection on indicatrix in (5.14) and using the fact that $h^i_{a|m} = 0$, we get

$$\alpha(p.H^i_{kh})_{|m|l} + \beta_l(p.H^i_{kh})_{|m} + \gamma_m(p.H^i_{kh})_{|l} + \nu_{lm}(p.H^i_{kh}) = 0. \tag{5.15}$$

Thus, we conclude

Theorem 5.3. *The projection of the $h(v)$-torsion tensor H^i_{kh} of an R^h-GR-F_n on indicatrix is generalized recurrent in the sense of Cartan.*

Let us consider a Finsler space whose deviation tensor H^i_h is *h-GR*. This space is characterized by (2.15). In view of (5.1a), the projection of the deviation tensor H^i_h on the indicatrix is given by

$$p.H^i_h = H^a_b h^i_a h^b_h. \tag{5.16}$$

Taking h-covariant derivative of (5.16) twice with respect to x^m and x^l successively and using the fact that $h^i_{a|m} = 0$, we get

$$(p.H^i_h)_{|m|l} = H^a_{b|m|l} h^i_a h^b_h. \tag{5.17}$$

Transvecting (5.17) by α and using (2.15), we get

$$\alpha(p.H^i_h)_{|m|l} = -(\beta_l H^a_{b|m} + \gamma_m H^a_{b|l} + \nu_{lm} H^a_b) h^i_a h^b_h. \tag{5.18}$$

Using the definition of projection on indicatrix in (5.18) and the fact that $h^i_{a|m} = 0$, we get

$$\alpha(p.H^i_h)_{|m|l} + \beta_l(p.H^i_h)_{|m} + \gamma_m(p.H^i_h)_{|l} + \nu_{lm}(p.H^i_h) = 0. \tag{5.19}$$

Thus, we conclude

Theorem 5.4. *The projection of the deviation tensor H^i_h of an R^h-GR-F_n on indicatrix is generalized recurrent in the sense of Cartan.*

Let us consider a Finsler space F_n for which the projection of the Cartan third curvature tensor R^i_{jkh} on indicatrix is generalized recurrent with respect to Cartan's connection. This space is characterized by (5.6).

Using the definition of projection on indicatrix in (5.6), we get

$$(5.20)\qquad \alpha(R^a_{bcd}h^i_a h^b_j h^c_k h^d_h)_{|m|l} + \beta_l(R^a_{bcd}h^i_a h^b_j h^c_k h^d_h)_{|m}$$

$$+\gamma_m(R^a_{bcd}h^i_a h^b_j h^c_k h^d_h)_{|l} + \nu_{lm}(R^a_{bcd}h^i_a h^b_j h^c_k h^d_h) = 0.$$

Using (5.1b) in (5.20), we get

$$(5.21)\qquad \alpha[R^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)]_{|m|l}$$

$$+\beta_l[R^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)]_{|m}$$

$$+\gamma_m[R^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)]_{|l}$$

$$+\nu_{lm}[R^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)] = 0.$$

Using (5.1a) and (1.8.8c) in (5.21), we get

$$(5.22)\qquad \alpha[R^i_{jkh} - R^i_{jkd}l^d l_h - R^i_{jch}l^c l_k + R^i_{jcd}l^c l_k l^d l_h - \frac{1}{F}H^i_{kh}l_j$$

$$+\frac{1}{F}H^i_{kd}l_j l^d l_h + \frac{1}{F}H^i_{ch}l_j l^c l_k - \frac{1}{F}H^i_{cd}l_j l^c l_k l^d l_h$$

$$- R^a_{jkh}l^i l_a + R^a_{jkd}l^i l_a l^d l_h + R^a_{jch}l^i l_a l^c l_k - R^a_{jcd}l^i l_a l^c l_k l^d l_h$$

$$+\frac{1}{F}H^a_{kh}l^il_al_j-\frac{1}{F}H^a_{kd}l^il_al_jl^dl_h-\frac{1}{F}H^a_{ch}l^il_al_jl^cl_k$$

$$+\frac{1}{F}H^a_{cd}l^il_al_jl^cl_kl^dl_h]_{|m||l}+\beta_l[R^i_{jkh}-R^i_{jkd}l^dl_h$$

$$-R^i_{jch}l^cl_k+R^i_{jcd}l^cl_kl^dl_h-\frac{1}{F}H^i_{kh}l_j+\frac{1}{F}H^i_{kd}l_jl^dl_h$$

$$+\frac{1}{F}H^i_{ch}l_jl^cl_k-\frac{1}{F}H^i_{cd}l_jl^cl_kl^dl_h-R^a_{jkh}l^il_a$$

$$+R^a_{jkd}l^il_al^dl_h+R^a_{jch}l^il_al^cl_k-R^a_{jcd}l^il_al^cl_kl^dl_h$$

$$+\frac{1}{F}H^a_{kh}l^il_al_j-\frac{1}{F}H^a_{kd}l^il_al_jl^dl_h-\frac{1}{F}H^a_{ch}l^il_al_jl^cl_k$$

$$+\frac{1}{F}H^a_{cd}l^il_al_jl^cl_kl^dl_h]_{|m}+\gamma_m[R^i_{jkh}-R^i_{jkd}l^dl_h$$

$$-R^i_{jch}l^cl_k+R^i_{jcd}l^cl_kl^dl_h-\frac{1}{F}H^i_{kh}l_j+\frac{1}{F}H^i_{kd}l_jl^dl_h$$

$$+\frac{1}{F}H^i_{ch}l_jl^cl_k-\frac{1}{F}H^i_{cd}l_jl^cl_kl^dl_h-R^a_{jkh}l^il_a+R^a_{jkd}l^il_al^dl_h$$

$$+R^a_{jch}l^il_al^cl_k-R^a_{jcd}l^il_al^cl_kl^dl_h+\frac{1}{F}H^a_{kh}l^il_al_j$$

$$-\frac{1}{F}H^a_{kd}l^il_al_jl^dl_h-\frac{1}{F}H^a_{ch}l^il_al_jl^cl_k+\frac{1}{F}H^a_{cd}l^il_al_jl^cl_kl^dl_h]_{||l}$$

$$+\nu_{lm}[R^i_{jkh}-R^i_{jkd}l^dl_h-R^i_{jch}l^cl_k+R^i_{jcd}l^cl_kl^dl_h-\frac{1}{F}H^i_{kh}l_j$$

$$+\frac{1}{F}H^i_{kd}l_jl^dl_h+\frac{1}{F}H^i_{ch}l_jl^cl_k-\frac{1}{F}H^i_{ch}l_jl^cl_kl^dl_h-R^a_{jkh}l^il_a+R^a_{jkd}l^il_al^dl_h$$

$$+R^a_{jch}l^il_al^cl_k-R^a_{jcd}l^il_al^cl_kl^dl_h+\frac{1}{F}H^a_{kh}l^il_al_j-\frac{1}{F}H^a_{kd}l^il_al_jl^dl_h$$

$$-\frac{1}{F}H^a_{ch}l^il_al_jl^cl_k+\frac{1}{F}H^a_{cd}l^il_al_jl^cl_kl^dl_h]=0.$$

Using (1.6.1a), (1.6.1b), (1.9.7b), (1.9.7c) and (1.9.12a) in (5.22), we get

(5.23) $$\alpha[R^i_{jkh}-R^i_{jkd}l^dl_h-R^i_{jch}l^cl_k+R^i_{jcd}l^cl_kl^dl_h-\frac{1}{F}H^i_{kh}l_j$$

$$-\frac{1}{F^2}H^i_kl_jl_h+\frac{1}{F^2}H^i_hl_jl_k-R^a_{jkh}l^il_a+R^a_{jkd}l^il_al^dl_h$$

$$+R^a_{jch}l^il_al^cl_k-R^a_{jcd}l^il_al^cl_kl^dl_h]_{|m|l}$$

$$+\beta_l[R^i_{jkh}-R^i_{jkd}l^dl_h-R^i_{jch}l^cl_k+R^i_{jcd}l^cl_kl^dl_h-\frac{1}{F}H^i_{kh}l_j$$

$$-\frac{1}{F^2}H^i_kl_jl_h+\frac{1}{F^2}H^i_hl_jl_k-R^a_{jkh}l^il_a+R^a_{jkd}l^il_al^dl_h$$

$$+R^a_{jch}l^cl_kl^il_a-R^a_{jcd}l^il_al^dl_hl^cl_k]_{|m}+\gamma_m[R^i_{jkh}-R^i_{jkd}l^dl_h$$

$$-R^i_{jch}l^cl_k+R^i_{jcd}l^cl_kl^dl_h-\frac{1}{F}H^i_{kh}l_j-\frac{1}{F^2}H^i_kl_jl_h$$

$$+\frac{1}{F^2}H^i_hl_jl_k-R^a_{jkh}l^il_a+R^a_{jkd}l^il_al^dl_h+R^a_{jch}l^cl_kl^il_a$$

$$-R^a_{jcd}l^il_al^dl_hl^cl_k]_{|l}+\nu_{lm}[R^i_{jkh}-R^i_{jkd}l^dl_h-R^i_{jch}l^cl_k$$

$$+R^{i}_{jcd}l^{c}l_{k}l^{d}l_{h}-\frac{1}{F}H^{i}_{kh}l_{j}-\frac{1}{F^{2}}H^{i}_{k}l_{j}l_{h}+\frac{1}{F^{2}}H^{i}_{h}l_{j}l_{k}$$

$$-R^{a}_{jkh}l^{i}l_{a}+R^{a}_{jkd}l^{i}l_{a}l^{d}l_{h}+R^{a}_{jch}l^{c}l_{k}l^{i}l_{a}-R^{a}_{jcd}l^{i}l_{a}l^{d}l_{h}l^{c}l_{k}]=0.$$

Suppose that the $h(v)$-torsion tensor H^{i}_{kh} is generalized recurrent. Then we have

$$\text{(A)} \qquad \alpha H^{i}_{kh|m|l}+\beta_{l}H^{i}_{kh|m}+\gamma_{m}H^{i}_{kh|l}+\nu_{lm}H^{i}_{kh}=0.$$

Transvecting (A) by y^{k} and using (1.6.11a) and (1.9.7b), we get

$$\text{(B)} \qquad \alpha H^{i}_{h|m|l}+\beta_{l}H^{i}_{h|m}+\gamma_{m}H^{i}_{h|l}+\nu_{lm}H^{i}_{h}=0 .$$

Using (A), (B), (1.6.11b), (1.6.11c) and the fact that R^{i}_{jkh} is skew-symmetric in its last two lower indices in the equation (5.23), we have

$$\text{(5.24)} \qquad \alpha R^{i}_{jkh|m|l}-\alpha(R^{i}_{jkd}l^{d}l_{h}+R^{i}_{jch}l^{c}l_{k}+R^{a}_{jkh}l^{i}l_{a}$$

$$-R^{a}_{jkd}l^{i}l_{a}l^{d}l_{h}-R^{a}_{jch}l^{i}l_{a}l^{c}l_{k})_{|m|l}+\beta_{l}R^{i}_{jkh|m}$$

$$-\beta_{l}(R^{i}_{jkd}l^{d}l_{h}+R^{i}_{jch}l^{c}l_{k}+R^{a}_{jkh}l^{i}l_{a}-R^{a}_{jkd}l^{i}l_{a}l^{d}l_{h}$$

$$-R^{a}_{jch}l^{i}l_{a}l^{c}l_{k})_{|m}+\gamma_{m}R^{i}_{jkh|l}-\gamma_{m}(R^{i}_{jkd}l^{d}l_{h}$$

$$+R^{a}_{jch}l^{c}l_{k}+R^{a}_{jkh}l^{i}l_{a}-R^{a}_{jkd}l^{i}l_{a}l^{d}l_{h}-R^{a}_{jch}l^{i}l_{a}l^{c}l_{k})_{|l}$$

$$+\nu_{lm}R^{i}_{jkh}-\nu_{lm}(R^{i}_{jkd}l^{d}l_{h}+R^{i}_{jch}l^{c}l_{k}+R^{a}_{jkh}l^{i}l_{a}$$

$$-R^{a}_{jkd}l^{i}l_{a}l^{d}l_{h}-R^{a}_{jch}l^{i}l_{a}l^{c}l_{k})=0.$$

In view of (1.6.1b), (1.8.9a) and (1.8.9c), the equation (5.24) may be written as

(5.25)
$$\alpha R^i_{jkh|m|l} - \alpha(R^i_{jkd}l^d l_h + R^i_{jch}l^c l_k - \frac{1}{F}R^a_{qkh}g_{ja}y^q l^i$$

$$+\frac{1}{F}R^a_{qkd}g_{ja}y^q l^i l^d l_h + \frac{1}{F}R^a_{qch}l^i g_{ja}y^q l^c l_k)_{|m|l}$$

$$+\beta_l R^i_{jkh|m} - \beta_l(R^i_{jkd}l^d l_h + R^i_{jch}l^c l_k - \frac{1}{F}R^a_{qkh}g_{ja}y^q l^i$$

$$+\frac{1}{F}R^a_{qkd}g_{ja}y^q l^i l_h l^d + \frac{1}{F}R^a_{qch}l^i g_{ja}y^q l^c l_k)_{|m}$$

$$+\gamma_m R^i_{jkh|l} - \gamma_m(R^i_{jkd}l^d l_h + R^i_{jch}l^c l_k - \frac{1}{F}R^a_{qkh}g_{ja}y^q l^i$$

$$+\frac{1}{F}R^a_{qkd}g_{ja}y^q l^i l_h l^d + \frac{1}{F}R^a_{qch}l^i g_{ja}y^q l^c l_k)_{|l}$$

$$+\nu_{lm}R^i_{jkh} - \nu_{lm}(R^i_{jkd}l^d l_h + R^i_{jch}l^c l_k - \frac{1}{F}R^a_{qkh}g_{ja}y^q l^i$$

$$+\frac{1}{F}R^a_{qkd}g_{ja}y^q l^i l_h l^d + \frac{1}{F}R^a_{qch}l^i g_{ja}y^q l^c l_k) = 0.$$

Using (1.6.1a), (1.8.8c) and (1.9.7b) in (5.25), we get

(5.26)
$$\alpha R^i_{jkh|m|l} - \alpha(R^i_{jkd}l^d l_h + R^i_{jch}l^c l_k - \frac{1}{F}l^i g_{ja}H^a_{kh}$$

$$-\frac{1}{F^2}l^i l_h g_{ja}H^a_k + \frac{1}{F^2}l^i l_k g_{ja}H^a_h)_{|m|l} + \beta_l R^i_{jkh|m}$$

$$-\beta_l(R^i_{jkd}l^d l_h + R^i_{jch}l^c l_k - \frac{1}{F}l^i g_{ja}H^a_{kh} - \frac{1}{F^2}l^i l_h g_{ja}H^a_k$$

$$+\frac{1}{F^2}l^i l_k g_{ja} H^a_h)_{|m} + \gamma_m R^i_{jkh|l} - \gamma_m (R^i_{jkd} l^d l_h$$

$$+ R^i_{jch} l^c l_k - \frac{1}{F} l^i g_{ja} H^a_{kh} - \frac{1}{F^2} l^i l_h g_{ja} H^a_k$$

$$+\frac{1}{F^2} l^i l_k g_{ja} H^a_h)_{|l} + v_{lm} R^i_{jkh} - v_{lm} (R^i_{jkd} l^d l_h$$

$$+ R^i_{jch} l^c l_k - \frac{1}{F} l^i g_{ja} H^a_{kh} - \frac{1}{F^2} l^i l_h g_{ja} H^a_k + \frac{1}{F^2} l^i l_k g_{ja} H^a_h) = 0.$$

In view of (1.6.11b), (1.6.11c), (1.6.10a), (1.6.1b), (A) and (B), the equation (5.26) may be written as

(5.27) $$\alpha R^i_{jkh|m|l} - \alpha (R^i_{jkd} l^d l_h + R^i_{jch} l^c l_k)_{|m|l} + \beta_l R^i_{jkh|m}$$

$$- \beta_l (R^i_{jkd} l^d l_h + R^i_{jch} l^c l_k)_{|m} + \gamma_m R^i_{jkh|l} - \gamma_m (R^i_{jkd} l^d l_h$$

$$+ R^i_{jch} l^c l_k)_{|l} + v_{lm} R^i_{jkh} - v_{lm} (R^i_{jkd} l^d l_h + R^i_{jch} l^c l_k) = 0.$$

In view of (1.6.1a), (1.8.9a) and (1.8.9c) the equation (5.27) becomes

(5.28) $$\alpha R^i_{jkh|m|l} - \alpha (\frac{1}{F} l_k g^{ip} R_{jpch} y^c - \frac{1}{F} l_h g^{ip} R_{jpdk} y^d)_{|m|l}$$

$$+ \beta_l R^i_{jkh|m} - \beta_l (\frac{1}{F} l_k g^{ip} R_{jpch} y^c - \frac{1}{F} l_h g^{ip} R_{jpdk} y^d)_{|m}$$

$$+ \gamma_m R^i_{jkh|l} - \gamma_m (\frac{1}{F} l_k g^{ip} R_{jpch} y^c - \frac{1}{F} l_h g^{ip} R_{jpdk} y^d)_{|l}$$

$$+ v_{lm} R^i_{jkh} - v_{lm} (\frac{1}{F} l_k g^{ip} R_{jpch} y^c - \frac{1}{F} l_h g^{ip} R_{jpdk} y^d) = 0.$$

Using $(R_{ijhk} - R_{hkij})y^h = C_{ikr}H^r_j - C_{jkr}H^r_i$ [120] in (5.28), we get

(5.29) $$\alpha R^i_{jkh|l|m|l'} - \alpha[\frac{1}{F}l_k g^{ip}(R_{chjp}y^c + C_{jhq}H^q_p$$

$$- C_{hpq}H^q_j) - \frac{1}{F}l_h g^{ip}(R_{dkjp}y^d + C_{jkq}H^q_p - C_{pkq}H^q_j)]_{|m|l'}$$

$$+ \beta_l R^i_{jkh|m} - \beta_l[\frac{1}{F}l_k g^{ip}(R_{chjp}y^c + C_{jhq}H^q_p - C_{hpq}H^q_j)$$

$$- \frac{1}{F}l_h g^{ip}(R_{dkjp}y^d + C_{jkq}H^q_p - C_{pkq}H^q_j)]_{|m} + \gamma_m R^i_{jkh|l'}$$

$$- \gamma_m[\frac{1}{F}l_k g^{ip}(R_{chjp}y^c + C_{jhq}H^q_p - C_{hpq}H^q_j)$$

$$- \frac{1}{F}l_h g^{ip}(R_{dkjp}y^d + C_{jkq}H^q_p - C_{pkq}H^q_j)]_{|l'} + \nu_{lm}R^i_{jkh}$$

$$- \nu_{lm}[\frac{1}{F}l_k g^{ip}(R_{chjp}y^c + C_{jhq}H^q_p - C_{hpq}H^q_j)$$

$$- \frac{1}{F}l_h g^{ip}(R_{dkjp}y^d + C_{jkq}H^q_p - C_{pkq}H^q_j)] = 0.$$

Suppose that the $(h)hv$-torsion tensor C_{ijk} is generalized recurrent in the space considered, then we have

(C) $$\alpha C_{ijk|m|l'} + \beta_l C_{ijk|m} + \gamma_m C_{ijk|l'} + \nu_{lm}C_{ijk} = 0.$$

Using (1.6.10b), (1.6.11b), (1.6.11c), (B) and (C) in (5.29), we have

(5.30) $$\alpha R^i_{jkh|m|l'} - \alpha(\frac{1}{F}l_k g^{ip}R_{chjp}y^c - \frac{1}{F}l_h g^{ip}$$

$$R_{dkjp}y^{d})_{|ml l}+\beta_{l}R^{i}_{jkh|lm}-\beta_{l}(\frac{1}{F}l_{k}g^{ip}R_{chjp}y^{c}$$

$$-\frac{1}{F}l_{h}g^{ip}R_{dkjp}y^{d})_{|m}+\gamma_{m}R^{i}_{jkh|l}-\gamma_{m}(\frac{1}{F}l_{k}g^{ip}R_{chjp}y^{c}$$

$$-\frac{1}{F}l_{h}g^{ip}R_{dkjp}y^{d})=0\ .$$

In view of (1.8.9a), (5.30) may be written as

$$\alpha R^{i}_{jkh|lm|l}-\alpha(\frac{1}{F}l_{k}g^{ip}g_{hq}R^{q}_{cjp}y^{c}-\frac{1}{F}l_{h}g^{ip} \tag{5.31}$$

$$g_{kq}R^{q}_{djp}y^{d})_{|ml}+\beta_{l}R^{i}_{jkh|lm}-\beta_{l}(\frac{1}{F}l_{k}g^{ip}g_{hq}R^{q}_{cjp}y^{c}$$

$$-\frac{1}{F}l_{h}g^{ip}g_{kq}R^{q}_{djp}y^{d})_{|m}+\gamma_{m}R^{i}_{jkh|l}-\gamma_{m}(\frac{1}{F}l_{k}g^{ip}$$

$$g_{hq}R^{q}_{cjp}y^{c}-\frac{1}{F}l_{h}g^{ip}g_{kq}R^{q}_{djp}y^{d})_{|l}+\nu_{lm}R^{i}_{jkh}$$

$$-\nu_{lm}(\frac{1}{F}l_{k}g^{ip}g_{hq}R^{q}_{cjp}y^{c}-\frac{1}{F}l_{h}g^{ip}g_{kq}R^{q}_{djp}y^{d})=0\,.$$

Using (1.8.8c) in (5.31), we get

$$\alpha R^{i}_{jkh|lm|l}-\alpha(\frac{1}{F}l_{k}g^{ip}g_{hq}H^{q}_{jp}-\frac{1}{F}l_{h}g^{ip}g_{kq}H^{q}_{jp})_{|ml} \tag{5.32}$$

$$+\beta_{l}R^{i}_{jkh|lm}-\beta_{l}(\frac{1}{F}l_{k}g^{ip}g_{hq}H^{q}_{jp}-\frac{1}{F}l_{h}g^{ip}g_{kq}H^{q}_{jp})_{|m}$$

$$+\gamma_{m}R^{i}_{jkh|l}-\gamma_{m}(\frac{1}{F}l_{k}g^{ip}g_{hq}H^{q}_{jp}-\frac{1}{F}l_{h}g^{ip}g_{kq}H^{q}_{jp})_{|l}$$

$$+v_{lm}R^{i}_{jkh}-v_{lm}\left(\frac{1}{F}l_k g^{ip}g_{hq}H^{q}_{jp}-\frac{1}{F}l_h g^{ip}g_{kq}H^{q}_{jp}\right)=0.$$

In view of (1.6.10a), (1.6.10b), (1.6.11b), (1.6.11c) and (A), we may write (5.32) as

$$\alpha R^{i}_{jkh|m|l}+\beta_l R^{i}_{jkh|m}+\gamma_m R^{i}_{jkh|l}+v_{lm}R^{i}_{jkh}=0. \tag{5.33}$$

Therefore, the Cartan third curvature tensor R^{i}_{jkh} is generalized recurrent. Thus, we conclude

Theorem 5.5. *If the projection of the Cartan third curvature tensor R^{i}_{jkh} on indicatrix is generalized recurrent, then the space is an R^h-GR-F_n characterized by (2.4) provided H^{i}_{kh} and C_{ijk} are h-GR in the sense of Cartan.*

Let us consider a Finsler space F_n for which the projection of the $h(v)$-torsion tensor H^{i}_{kh} on indicatrix is generalized recurrent with respect to Cartan's connection characterized by (5.15). Using the definition of projection on indicatrix in (5.15), we get

$$\alpha(H^{a}_{bc}h^{i}_{a}h^{b}_{k}h^{c}_{h})_{|m|l}+\beta_l(H^{a}_{bc}h^{i}_{a}h^{b}_{k}h^{c}_{h})_{|m} \tag{5.34}$$

$$+\gamma_m(H^{a}_{bc}h^{i}_{a}h^{b}_{k}h^{c}_{h})_{|l}+v_{lm}(H^{a}_{bc}h^{i}_{a}h^{b}_{k}h^{c}_{h})=0.$$

Using (5.1b) in (5.34), we get

$$\alpha[H^{a}_{bc}(\delta^{i}_{a}-l^{i}l_a)(\delta^{b}_{k}-l^{b}l_k)(\delta^{c}_{h}-l^{c}l_h)]_{|m|l} \tag{5.35}$$

$$+\beta_l[H^{a}_{bc}(\delta^{i}_{a}-l^{i}l_a)(\delta^{b}_{k}-l^{b}l_k)(\delta^{c}_{h}-l^{c}l_h)]_{|m}$$

$$+\gamma_m[H^{a}_{bc}(\delta^{i}_{a}-l^{i}l_a)(\delta^{b}_{k}-l^{b}l_k)(\delta^{c}_{h}-l^{c}l_h)]_{|l}$$

$$+v_{lm}[H^{a}_{bc}(\delta^{i}_{a}-l^{i}l_a)(\delta^{b}_{k}-l^{b}l_k)(\delta^{c}_{h}-l^{c}l_h)]=0.$$

Using (1.6.1a), (1.6.1b), (1.9.7b), (1.9.7c) and (1.9.12a) in (5.35), we get

(5.36) $$\alpha(H^i_{kh} - H^i_{kc}l^c l_h - H^i_{bh}l^b l_k)_{|m|l} + \beta_l(H^i_{kh} - H^i_{kc}l^c l_h - H^i_{bh}l^b l_k)_{|m} + \gamma_m(H^i_{kh} - H^i_{kc}l^c l_h - H^i_{bh}l^b l_k)_{|l} + \nu_{lm}(H^i_{kh} - H^i_{kc}l^c l_h - H^i_{bh}l^b l_k) = 0.$$

Using (1.6.1a) and (1.9.7b) in (5.36), we get

(5.37) $$\alpha(H^i_{kh} + \frac{1}{F}H^i_k l_h - \frac{1}{F}H^i_h l_k)_{|m|l} + \beta_l(H^i_{kh} + \frac{1}{F}H^i_k l_h - \frac{1}{F}H^i_h l_k)_{|m} + \gamma_m(H^i_{kh} + \frac{1}{F}H^i_k l_h - \frac{1}{F}H^i_h l_k)_{|l} + \nu_{lm}(H^i_{kh} + \frac{1}{F}H^i_k l_h - \frac{1}{F}H^i_h l_k) = 0.$$

In view of (1.6.11b) and (1.6.11c), the equation (5.37) may be written as

(5.38) $$\alpha H^i_{kh|m|l} + \beta_l H^i_{kh|m} + \gamma_m H^i_{kh|l} + \nu_{lm}H^i_{kh} + \frac{1}{F}l_h(\alpha H^i_{k|m|l} + \beta_l H^i_{k|m} + \gamma_m H^i_{k|l} + \nu_{lm}H^i_k) - \frac{1}{F}l_k(\alpha H^i_{h|m|l} + \beta_l H^i_{h|m} + \gamma_m H^i_{h|l} + \nu_{lm}H^i_h) = 0.$$

Now if H^i_h is *h-GR* in the sense of Cartan the equation (5.38) may be written as

(5.39) $$\alpha H^i_{kh|m|l} + \beta_l H^i_{kh|m} + \gamma_m H^i_{kh|l} + \nu_{lm}H^i_{kh} = 0\ .$$

Therefore the $h(v)$-torsion tensor H^i_{kh} is generalized recurrent in the sense of Cartan. Thus, we conclude

Theorem 5.6. *If the projection of the $h(v)$-torsion tensor H^i_{kh} on indicatrix is generalized recurrent then the space is an R^h-GR-F_n characterized by* (2.14) *provided H^i_h is h-GR in the sense of Cartan.*

Let us consider a Finsler space F_n for which the projection of the deviation tensor H^i_h on indicatrix is generalized recurrent with respect to Cartan's connection characterized by (5.19). Using the definition of projection on indicatrix in (5.19), we get

$$\alpha(H^a_b h^i_a h^b_h)_{|m|l} + \beta_l (H^a_b h^i_a h^b_h)_{|m} + \gamma_m (H^a_b h^i_a h^b_h)_{|l} + v_{lm}(H^a_b h^i_a h^b_h) = 0 . \tag{5.40}$$

Using (5.1b) in (5.40), we get

$$\alpha[H^a_b(\delta^i_a - l^i l_a)(\delta^b_h - l^b l_h)]_{|m|l} + \beta_l [H^a_b(\delta^i_a - l^i l_a)(\delta^b_h - l^b l_h)]_{|m} \tag{5.41}$$

$$+ \gamma_m [H^a_b(\delta^i_a - l^i l_a)(\delta^b_h - l^b l_h)]_{|l} + v_{lm}[H^a_b(\delta^i_a - l^i l_a)(\delta^b_h - l^b l_h)] = 0 .$$

Using (1.6.1a), (1.6.1b), (1.9.12b) in (5.41), we get

$$\alpha H^i_{h|m|l} + \beta_l H^i_{h|m} + \gamma_m H^i_{h|l} + v_{lm} H^i_h = 0 .$$

Therefore the deviation tensor H^i_h is *h-GR*.

Thus, we conclude

Theorem 5.7. *If the projection of the deviation tensor H^i_h on indicatrix is generalized recurrent then the space is an R^h-GR-F_n characterized by* (2.15).

* * * * *

Chapter III

CERTAIN TYPES OF PROJECTIVE MOTION

1. Introduction

The projective motions generated by contra, special concircular, recurrent, concircular, torseforming and birecurrent vector fields have been discussed in Riemannian and non-Riemannian spaces of recurrent curvature by K. Takano [133], S. Yamaguchi [145], K. Amur and P. Desai [6], T. Adati and T. Miyazawa [3] and others. Various results of these authors have been extended to Finsler spaces of recurrent curvature and some other special Finsler spaces by R. S. Sinha [129, 130], R. S. Sinha and S. A. Faruqui [131], R. B. Misra [61], R. B. Misra, N. Kishore and P. N. Pandey [62], R. B. Misra and F. M. Meher [64], T. Aikou [4], M. Hashiguchi [25], H. Izumi [29, 30, 31], B. N. Prasad, B. N. Gupta and D. D. Singh [117], B. N. Prasad, V. P. Singh and J. N. Singh [118], D. N. Yablokov [144], P. N. Pandey and R. Verma [115], P. N. Pandey and R. B. Misra [112] and others. The results obtained by these authors were concerned with special type of Finsler spaces and could not throw any light on the behaviour of these results in a general Finsler space. P. N. Pandey [93, 94, 97, 98, 102] discussed this problem and extended the result of K. Yano and T. Nagano [134] and generalized several results in this direction. Fahmi Yaseen Abdo Qasem [149] discussed infinitesimal affine motions generated by vector fields satisfying some generalized conditions than the above authors.

The aim of this chapter is to discuss infinitesimal projective motions generated by the vector fields satisfying the generalized conditions considered by Fahmi Yaseen Abdo Qasem.

2. Projective Motion

An infinitesimal transformation generated by a vector field $v^i(x^j)$ is a projective

motion if and only if the Lie-derivative of connection coefficients is given by (1.11.7c).

Also an infinitesimal transformation is an affine motion if and only if $\pounds G^i_{jk} = 0$. Therefore a projective motion is an affine motion if P vanishes.

The vector field $v^i(x^j)$ is called contra, concurrent, special concircular, recurrent, concircular, torseforming and birecurrent if it satisfies

(2.1) a) $\mathcal{B}_k v^i = 0$,

b) $\mathcal{B}_k v^i = c\delta^i_k$, c being constant

c) $\mathcal{B}_k v^i = \rho\delta^i_k$, ρ is not constant

d) $\mathcal{B}_k v^i = \mu_k v^i$,

e) $\mathcal{B}_k v^i = \mu_k v^i + \rho\delta^i_k$, $\mathcal{B}_j \mu_k = \mathcal{B}_k \mu_j$

f) $\mathcal{B}_k v^i = \mu_k v^i + \rho\delta^i_k$

and

g) $\mathcal{B}_j \mathcal{B}_k v^i = \mu_{jk} v^i$

respectively. The projective motion generated by above vectors is called a special concircular projective motion, a concircular projective motion, a recurrent projective motion, a torseforming projective motion and a birecurrent projective motion according as the generating vector is a special concircular vector field, a concircular vector field, a recurrent vector field, a torseforming vector field or a birecurrent vector field.

We shall discuss the projective motion generated by the vector fields satisfying more general conditions viz.,

(2.2.) (i) $\mathcal{B}_j\mathcal{B}_k v^i = 0,$

(ii) $\mathcal{B}_j\mathcal{B}_k v^i = \rho_j \delta^i_k,$

(iii) $\mathcal{B}_j\mathcal{B}_k v^i = \rho_j \delta^i_k + \mu_k \delta^i_j,$

(iv) $\mathcal{B}_j\mathcal{B}_k v^i = \mu_{jk} v^i + a_{jk} y^i,$

(v) $\mathcal{B}_j\mathcal{B}_k v^i = a_{jk} v^i + \mu_j \delta^i_k.$

3. Special Projective Motion Case (i)

We know that every affine motion is projective motion and Fahmi Yaseen Abdo Qasem [149] proved that every vector field satisfying (2.2i) generates an affine motion. Hence every vector field satisfying (2.2i) generates a projective motion.

He also proved that if a vector field $v^i(x^j)$ satisfying (2.2i) generates an infinitesimal transformation in a recurrent or birecurrent space, then the vector field $v^i(x^j)$ is orthogonal to the recurrence vector or $a_{lm}v^m = 0$.

4. Special Projective Motion Case (ii)

Let us consider a Finsler space F_n admitting a special projective motion characterized by (1.11.7c) and (2.2ii). In view of (1.11.7c) and (2.2ii), (1.11.5b) may be written as

(4.1) $$\rho_j \delta^i_k + H^i_{kjh} v^h + G^i_{jkh} \mathcal{B}_r v^h y^r = y^i P_{jk} + \delta^i_j P_k + \delta^i_k P_j .$$

Differentiating (2.2ii) partially with respect to y^m and using the commutation formula (1.7.10), we have

(4.2) $$\mathcal{B}_j(G^i_{mkr}v^r)+G^i_{mjr}\ \mathcal{B}_k v^r - G^r_{mjk}\ \mathcal{B}_r v^i = (\dot{\partial}_m\, \rho_j)\delta^i_k .$$

Transvecting (4.2) by y^k and using (1.7.7), we get

(4.3) $$G^i_{mjr}y^k\,\mathcal{B}_k v^r = y^i(\dot{\partial}_m\, \rho_j).$$

In view of (4.3), we may write (4.1) as

(4.4) $$\rho_j\delta^i_k + H^i_{kjh}v^h + y^i(\dot{\partial}_j\, \rho_k) = y^i P_{jk} + \delta^i_k P_j + \delta^i_j P_k .$$

Transvecting (4.4) by y^k and using equations (1.9.2b), (1.10.8) and $G^i_{mjr}y^j y^k$ $\mathcal{B}_k v^r = y^i y^j\, \dot{\partial}_m\, \rho_j = 0$ (which is direct consequences of (4.3)), we have

(4.5) $$\rho_j y^i + H^i_{jh}v^h = \delta^i_j P + y^i P_j .$$

Transvecting (4.5) by v^j and using the skew-symmetry of H^i_{jh}, we have

(4.6) $$(P_j v^j - \rho_j v^j)y^i + v^i(-P) = 0 .$$

In view of the lemma established by P. N. Pandey [94] which states that $av^i + by^i = 0$ implies $a = b = 0$, the equation (4.6) implies

(4.7) a) $P_j v^j = \rho_j v^j$ and b) $P=0$.

Since $P = 0$ implies $P_j = 0$, (4.7a) may be written as $\rho_j v^j = 0$.

We know that a projective motion is an affine motion if P vanishes identically. Hence from (4.7b) the projective motion considered is an affine motion. But Fahmi Yaseen Abdo Qasem [149] proved that an infinitesimal transformation generated by a vector field $v^i(x^j)$ satisfying (2.2ii) can not be an affine motion in a Finsler space. This leads to a contradiction. Thus we have

Theorem 4.1. *A vector field* $v^i(x^j)$ *characterized by* (2.2ii) *can not generate a projective motion in a Finsler space.*

5. Special Projective Motion Case (iii)

Let us consider a Finsler space F_n admitting a special projective motion characterized by (2.2.iii) and (1.11.7c). In view of (1.11.7c) and (2.2iii), (1.11.5b) may be written as

$$(5.1)\qquad \rho_j\delta^i_k+\mu_k\delta^i_j+H^i_{kjm}v^m+G^i_{jkm}\mathcal{B}_r v^m y^r = y^i P_{jk}+\delta^i_j P_k+\delta^i_k P_j .$$

Differentiating (2.2iii) partially with respect to y^m and using the commutation formula (1.7.10), we get

$$(5.2)\qquad \mathcal{B}_j(G^i_{mkr}v^r)+G^i_{mjr}\mathcal{B}_k v^r - G^i_{mjk}\mathcal{B}_r v^i = (\partial_m \rho_j)\delta^i_k+(\partial_m \mu_k)\delta^i_j .$$

Transvecting (5.2) by y^k and using the equation (1.7.7), we get

$$(5.3)\qquad G^i_{mjr}\mathcal{B}_k v^r y^k = y^i(\partial_m \rho_j)+y^k(\partial_m \mu_k)\delta^i_j .$$

In view of (5.3), we may write (5.1) as

$$(5.4)\quad \rho_j\delta^i_k+\mu_k\delta^i_j+H^i_{kjm}v^m+y^i(\dot\partial_j \rho_k)+y^r(\dot\partial_j \mu_r)\delta^i_k=\delta^i_j P_k+\delta^i_k P_j+y^i P_{jk} .$$

Transvecting (5.4) by y^k and using equations (1.9.2b), (1.10.8) and $y^i y^k(\dot{\partial}_j \rho_k) + y^r y^i(\dot{\partial}_j \mu_r) = 0$ (which is direct consequence of (5.3)), we have

(5.5) $$\rho_j y^i + \mu_k y^k \delta^i_j + H^i_{jm} v^m = \delta^i_j P + P_j y^i .$$

Transvecting (5.5) by v^j and using the skew-symmetry of H^i_{jm}, we have

(5.6) $$(\rho_j v^j - P_j v^j) y^i + v^i(\mu_k y^k - P) = 0 .$$

In view of the lemma established by P. N. Pandey [94] which states that $av^i + by^i = 0$ implies $a = b = 0$, the equation (5.6) implies

(5.7) a) $\rho_j v^j = P_j v^j$ and b) $\mu_k y^k = P$.

Using (5.7.b) in (5.5) and transvecting this equation by y_i and using equation (1.4.4a) and (1.9.7b), we get

(5.8) $$F^2(\rho_j - P_j) = 0 .$$

Since F is the metric function which does not vanish, we have

(5.9) $$\rho_j = P_j .$$

In view of equation (5.9) and (5.7b), we may write (5.5) as

(5.10) a) $$H^i_{jm} v^m = 0 .$$

which implies

(5.10) b) $$H^i_{kjm} v^m = 0 .$$

From equation (5.10a), (5.9) and (5.7), we may write (5.4) as

$$\mu_k \delta^i_j + (y^r \dot{\partial}_j \mu_r)\delta^i_k = \delta^i_j P_k,$$

which may be written as

$$\mu_k \delta^i_j - \mu_j \delta^i_k + \rho_j \delta^i_k - \rho_k \delta^i_j = 0. \tag{5.11}$$

Contracting the indices i and j in equation (5.11), we get

$$\mu_k = \rho_k. \tag{5.12}$$

Thus, we may conclude

Theorem 5.1. *A vector field* $v^i(x^j)$ *characterized by* (2.2iii) *which generates a projective motion necessarily satisfies* (5.7), (5.9), (5.10) *and* (5.12).

Let us consider a recurrent Finsler space $F_n (n > 2)$ admitting a special projective motion characterized by (1.11.7c) and (2.2iii). We have the identity [89].

$$\lambda_m H^i_{jk} + \lambda_k H^i_{mj} + \lambda_j H^i_{km} = 0, \tag{5.13}$$

for a recurrent Finsler space whose recurrence vector is λ_m. Transvecting (5.13) by v^m and using (5.10a), we get

$$\lambda_m v^m H^i_{jk} = 0. \tag{5.14}$$

Equation (5.14) implies $\lambda_m v^m = 0$, for $H^i_{jk} \neq 0$ (the vanishing of the tensor H^i_{jk} implies the vanishing of the curvature tensor H^i_{jkh}).

Thus, we have

Theorem 5.2. *If a vector field* $v^i(x^j)$ *satisfying* (2.2iii) *generates a projective motion in a recurrent Finsler space, then the vector field* $v^i(x^j)$ *is orthogonal to the recurrence vector i.e.* $\lambda_m v^m = 0$.

Let us consider a birecurrent Finsler space admitting a special projective motion characterized by (1.11.7c) and (2.2iii). A birecurrent Finsler space F_n with recurrence tensor η_{lm} satisfies the identity [73]

(5.15) $$\eta_{lm} H^i_{jk} + \eta_{lk} H^i_{mj} + \eta_{lj} H^i_{km} = 0 .$$

Transvecting (5.15) by v^m and using (5.10a), we get

(5.16) $$\eta_{lm} v^m H^i_{jk} = 0 .$$

Equation (5.16) implies $\eta_{lm} v^m = 0$, for $H^i_{jk} \neq 0$ (the vanishing of the tensor H^i_{jk} implies the vanishing of the curvature tensor H^i_{jkh}).

Thus, we have

Theorem 5.3. *If a vector field* $v^i(x^j)$ *satisfying* (2.2iii) *generates a projective motion in a birecurrent Finsler space, then the vector field* $v^i(x^j)$ *is orthogonal to the recurrence tensor i.e.* $\eta_{lm} v^m = 0$.

6. Special Projective Motion Case (iv)

Let us consider a Finsler space F_n admitting a special projective motion characterized by (1.11.7c) and (2.2iv). In view of (1.11.7c) and (2.2iv), (1.11.5b) may be written as

(6.1) $$\mu_{jk} v^i + a_{jk} y^i + H^i_{kjm} v^m + G^i_{jkm} \mathcal{B}_r v^m y^r = y^i P_{jk} + \delta^i_j P_k + \delta^i_k P_j .$$

Differentiating (2.2iv) partially with respect to y^m and using the commutation formula (1.7.10), we get

(6.2) $$\mathcal{B}_j G^i_{mkr} v^r + G^i_{mjr} \mathcal{B}_k v^r - G^r_{mjk} \mathcal{B}_r v^i = (\dot{\partial}_m \mu_{jk}) v^i + (\dot{\partial}_m a_{jk}) y^i + a_{jk} \delta^i_m .$$

Transvecting (6.2) by y^k and using equation (1.7.7), we get

(6.3) $$G^i_{mjr} y^k \mathcal{B}_k v^r = y^k (\dot{\partial}_m \mu_{jk}) v^i + y^i y^k \dot{\partial}_m a_{jk} + y^k a_{jk} \delta^i_m .$$

In view of (6.3), we may write (6.1) as

(6.4) $$\mu_{jk} v^i + a_{jk} y^i + H^i_{kjm} v^m + y^r (\dot{\partial}_j \mu_{kr}) v^i + y^i y^r (\dot{\partial}_j a_{kr})$$

$$+ y^r a_{kr} \delta^i_j = y^i P_{jk} + \delta^i_j P_k + \delta^i_k P_j .$$

Transvecting (6.4) by y^k and using equations (1.9.2b), (1.10.8) and $y^k y^r (\dot{\partial}_j \mu_{kr}) v^i + y^i y^r y^k \dot{\partial}_j a_{kr} + y^k y^r \delta^i_j a_{kr} = 0$ (which is a direct consequence of (6.3)), we have

(6.5) $$\mu_{jk} v^i y^k + a_{jk} y^i y^k + H^i_{jm} v^m = P \delta^i_j + P_j y^i .$$

Transvecting (6.5) by v^j and using the skew-symmetry of H^i_{jm}, we have

(6.6) $$(\mu_{jk} v^j y^k - P) v^i + (a_{jk} y^k v^j - P_j v^j) y^i = 0 .$$

In view of the lemma established by P. N. Pandey [94] which states that $a v^i + b y^i = 0$ implies $a = b = 0$, the equation (6.6) implies

(6.7) a) $\mu_{jk} v^j y^k = P$ and b) $a_{jk} y^k v^j = P_j v^j .$

Transvecting (6.5) by y_i and using (1.4.4a) and (1.9.2b), we get

$$\mu_{jk}v^i y_i y^k = F^2(P_j - a_{jk}y^k) + Py_j.$$

Since the vector v^i is independent of $y_i, y_i v^i \neq 0$. Therefore the above equation implies

$$\mu_{jk}y^k = \frac{F^2}{\alpha}(P_j - a_{jk}y^k) + \frac{P}{\alpha}y_j, \tag{6.8}$$

where $\alpha = y_i v^i$.

Let us assume

$$\mu_{jk}y^k = 0. \tag{6.9}$$

In view of (6.9), the condition (6.8) implies

$$F^2(P_j - a_{jk}y^k) + Py_j = 0. \tag{6.10}$$

Transvecting (6.10) by v^j and using (6.7b), we get

$$Py_j v^j = 0.$$

Since $y_j v^j \neq 0$, we find P =0 and hence the projective motion is an affine motion. Thus, we conclude

Theorem 6.1. *If the vector field* $v^i(x^j)$ *characterized by* (2.2iv) *generates a projective motion and* $\mu_{jk}y^k = 0$, *then the projective motion is an affine motion.*

Fahmi Yaseen Abdo Qasem [149] proved that if the vector field characterized by (2.2iv) generates an affine motion and $\mu_{jk}y^k = 0$, then μ_{jk} and a_{jk} are symmetric. In view of this theorem, we conclude

Theorem 6.2. *If the vector field* $v^i(x^j)$ *characterized by* (2.2iv) *generates a projective motion and* $\mu_{jk}y^k = 0$, *then* μ_{jk} *and* a_{jk} *are symmetric.*

Recurrent and birecurrent Finsler spaces are characterized by

(6.11) $$\mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh}$$

and

(6.12) $$\mathcal{B}_l \mathcal{B}_m H^i_{jkh} = \eta_{lm} H^i_{jkh}$$

respectively. In the above equation λ_m and η_{lm} are non-zero vector and tensor field. If the vector field $v^i(x^j)$ characterized by (2.2iv) and satisfying condition $\mu_{jk}y^k = 0$, generates an affine motion in these spaces, then the vector field $v^i(x^j)$ must satisfy the condition

(6.13) $$\lambda_m v^m = 0$$

and

(6.14) $$\eta_{lm} v^m = 0$$

according as the space is recurrent or birecurrent. Thus the condition (6.13) or (6.14) is necessary for a special affine motion satisfying (2.2iv) according as the space is recurrent or birecurrent. This result was proved by Fahmi Yaseen Abdo Qasem [149]. However these conditions are not sufficient for an infinitesimal transformation to be an affine motion. But they are sufficient for a special projective motion satisfying (2.2iv) and $\mu_{jk}y^k = 0$ to be an affine motion as may be seen from the following

Theorem 6.3. *The condition for orthogonality of a vector field* $v^i(x^j)$ *satisfying (2.2iv) and* $\mu_{jk}y^k = 0$ *with recurrence vector of a recurrent Finsler space* $F_n (n > 3)$ *is sufficient for the projective motion generated by the vector field* $v^i(x^j)$ *to be an affine motion.*

Proof.

Let us consider a recurrent Finsler space $F_n (n > 3)$ admitting a special projective motion characterized by (1.11.7c) and (2.2iv) and satisfying $\mu_{jk}y^k = 0$. We have seen that (6.5) is a natural consequence of (1.11.7c) and (2.2iv). We also have the identity [89]

(6.15) $$\lambda_m H^i_{jk} + \lambda_k H^i_{mj} + \lambda_j H^i_{km} = 0,$$

for a recurrent Finsler space whose recurrence vector is λ_m. Transvecting (6.15) by v^m and using (6.5), we get

(6.16) $$\lambda_m v^m H^i_{jk} - \lambda_k [P\delta^i_j + P_j y^i - \mu_{jh} v^i y^h - a_{jh} y^i y^h] + \lambda_j [P\delta^i_k + P_k y^i - \mu_{kh} v^i y^h - a_{kh} y^i y^h] = 0.$$

If the vector field $v^i(x^j)$ is orthogonal to the recurrence vector λ_m i.e. $\lambda_m v^m = 0$, the equation (6.16) reduces to

(6.17) $$\lambda_j [P\delta^i_k + P_k y^i - \mu_{kh} v^i y^h - a_{kh} y^i y^h] - \lambda_k [P\delta^i_j + P_j y^i - \mu_{jh} v^i y^h - a_{jh} y^i y^h] = 0.$$

Contracting (6.17) with respect to i and k and using (6.7) and $\lambda_k v^k = 0$, we have

(6.18) $$\lambda_j (nP - a_{kh} y^k y^h) - (\lambda_j P + P_j \lambda_k y^k - a_{jh} y^k y^h \lambda_k) = 0.$$

Transvecting (6.18) by y^j , we get $(n-2)P\lambda_k y^k = 0$. Since $n > 3$, we have at least one of the following conditions

(6.19) a) $P = 0,$ b) $\lambda_k y^k = 0$.

Suppose $\lambda_k y^k = 0$. The partial derivative of this equation with respect to y^j yields

$$(\dot{\partial}_j \lambda_k) y^k + \lambda_j = 0.$$

But the recurrence vector is independent of directional arguments i.e. $\dot{\partial}_j \lambda_k = 0$. Hence $\lambda_j = 0$, a contradiction. Therefore (6.19b) can not be true. Thus we have $P = 0$, i.e. the projective motion is an affine motion.

Theorem 6.4. *The condition for orthogonality of a vector field* $v^i(x^j)$ *satisfying* (2.2iv) *and* $\mu_{jk} y^k = 0$ *with the birecurrence vector field of a birecurrent Finsler space* $F_n (n > 3)$ *is sufficient for the projective motion generated by the vector field* $v^i(x^j)$ *to be an affine motion.*

Proof.

Let us consider a birecurrent Finsler space admitting a special projective motion characterized by (1.11.7c), (2.2iv) and $\mu_{jk} y^k = 0$. A birecurrent Finsler space F_n with recurrence tensor η_{lm} satsifies the identity [73]

(6.20) $$\eta_{lm} H^i_{jk} + \eta_{lk} H^i_{mj} + \eta_{lj} H^i_{km} = 0.$$

Transvecting (6.20) by v^m and using equation (6.5), we have

(6.21) $$\eta_{lm} v^m H^i_{jk} - \eta_{lk} (P\delta^i_j + P_j y^i - \mu_{jh} v^i y^h - a_{jh} y^i y^h)$$

$$+\eta_{lj}(P\delta^i_k + P_k y^i - \mu_{kh}v^i y^h - a_{kh}y^i y^h) = 0.$$

Contracting the indices i and j in the equation (6.21) and using (6.8), we get

(6.22) $$\eta_{lm}v^m H^i_{ik} - \eta_{lk}(nP - a_{ih}y^i y^h) + \eta_{li}(P_k y^i - a_{kh}y^i y^h) + \eta_{lk}P = 0.$$

If the vector field $v^i(x^j)$ satisfies the condition $\eta_{lm}v^m = 0$ equation (6.22) reduces to

(6.23) $$(n-1)P\eta_{lk} - y^i y^h(a_{ih}\eta_{lk} - a_{kh}\eta_{li}) - \eta_{li}y^i P_k = 0.$$

Transvecting (6.23) by y^k and using equation (1.10.8a) and $n > 3$, we have $P\eta_{lk}y^k = 0$, which implies at least one of the following

(6.24) a) $\eta_{lk}y^k = 0$ b) $P = 0.$

If $\eta_{lk}y^k = 0$, equation (6.23) implies

$$[(n-1)P - a_{ih}y^i y^h]\eta_{lk} = 0$$

which implies

(6.25) $$a_{ih}y^i y^h = (n-1)P$$

as $\eta_{lk} \neq 0$.

Transvecting (6.21) by y^k and using $\eta_{lm}v^m = 0, \mu_{kh}y^h = 0$, (6.24a) and (6.25), we get $(n-3)y^i P = 0$. Since $n > 3$ we, have $P = 0$. Thus the projective motion is an affine motion.

7. Special Projective Motion Case (v)

Let us consider a Finsler space F_n admitting a special projective motion characterized by, (2.2v) and (1.11.7c). In view of (1.11.7c) and (2.2v), (1.11.5b) may be written as

(7.1) $$a_{jk}v^i + \mu_j\delta^i_k + H^i_{kjm}v^m + G^i_{jkm}\mathcal{B}_r v^m y^r = y^i P_{jk} + \delta^i_j P_k + \delta^i_k P_j .$$

Differentiating (2.2v) partially with respect to y^m and using the commutation formula (1.7.10), we get

(7.2) $$\mathcal{B}_j G^i_{mkr}v^r + G^i_{mjr}\mathcal{B}_k v^r - G^r_{mjk}\mathcal{B}_r v^i = (\dot\partial_m a_{jk})v^i + (\dot\partial_m \mu_j)\delta^i_k .$$

Transvecting (7.2) by y^k and using equation (1.7.7), we get

(7.3) $$G^i_{mjr}y^k\mathcal{B}_k v^r = y^k(\dot\partial_m a_{jk})v^i + y^i(\dot\partial_m \mu_j).$$

In view of (7.3), we may write (7.1) as

(7.4) $$a_{jk}v^i + \mu_j\delta^i_k + H^i_{kjm}v^m + y^r(\dot\partial_j a_{kr})v^i + y^i(\dot\partial_j \mu_k)$$

$$= y^i P_{jk} + \delta^i_j P_k + \delta^i_k P_j .$$

Transvecting (7.4) by y^k and using equation (1.9.2b), (1.10.8) and $y^r y^k(\dot\partial_j a_{kr})v^i + y^k y^i(\dot\partial_j \mu_k) = 0$ (which is direct consequences of (7.3)), we have

(7.5) $$a_{jk}v^i y^k + \mu_j y^i + H^i_{jm}v^m = \delta^i_j P + y^i P_j .$$

Transvecting (7.5) by v^j and using the skew-symmetry of H^i_{jm}, we have

$$(a_{jk}v^j y^k - P)v^i + (\mu_j v^j - P_j v^j)y^i = 0. \tag{7.6}$$

In view of the lemma established by P. N. Pandey [94] which states that $av^i + by^i = 0$, implies $a = b = 0$, the equation (7.6) implies

(7.7) a) $a_{jk}v^j y^k = P$ and b) $\mu_j v^j = P_j v^j$.

Thus, we may conclude

Theorem 7.1. *The vector field* $v^i(x^j)$, *characterized by* (2.2v) *which generates a projective motion necessarily satisfies* (7.7).

Let us consider that the tensor a_{jk} is skew-symmetric. Transvecting (7.5) by y^j and using the skew-symmetry of a_{jk} and the equation (1.9.7b), we get

$$H^i_m v^m + \mu_j y^j y^i = 2y^i P. \tag{7.8}$$

Transvecting (7.8) by y_i and using equations (1.4.4a) and (1.9.7c), we have $F^2(2P - \mu_j y^j) = 0$. Since F is the metric function which does not vanish anywhere, we find

$$\mu_j y^j = 2P. \tag{7.9}$$

Contracting the indices i and j in (7.5) and using the equations (7.7a) and (7.9), we get

$$H_m v^m = (n-2)P. \tag{7.10}$$

Thus, we may conclude

Theorem 7.2. *A vector field characterized by (2.2v) with skew-symmetric tensor* a_{jk}, *which generates a projective motion necessarily satisfies* (7.9) *and* (7.10).

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Chapter IV

HYPERSURFACES OF SPECIAL FINSLER SPACES

1. Introduction

The study of the hypersurface of a recurrent space was initiated by A. Moór [73]. Miyazawa and Chuman [68] have studied umbilical subspaces of recurrent Riemannian spaces. The study of umbilical subspaces of a recurrent Riemannian space was extended to a recurrent Finsler space by Singh and Singh [125]. This study is based on Cartan's process of covariant differentiation. U. P. Singh and G. C. Chaubey [124] have studied the properties of umbilical hypersurfaces immersed in a Finsler space which is recurrent in the sense of Berwald. They have investigated conditions under which a hypersurface immersed in a recurrent Finsler space is recurrent.

A Finsler space whose torsion tensor is recurrent, called as a C^h-recurrent space, was introduced by Makoto Matsumoto [39] for the first time. Reema Verma [138] generalized the condition of Makoto Matsumoto and developed the theory of a C^h-birecurrent space. She discussed various properties of such space.

In this chapter we define a C^δ-recurrent and a C^δ-birecurrent Finsler space and study some properties of such spaces. Some results concerning a totally geodesic and umbilical hypersurface of C^δ-recurrent and C^δ-birecurrent spaces have been obtained. The hypersurfaces of C^h-recurrent and C^h-birecurrent Finsler spaces will also be discussed in this chapter.

In the last section of this chapter we study the properties of a hypersurface of a C2-like Finsler space.

2. A Hypersurface of a Finsler Space

Let F_n be a Finsler space of dimension n with fundamental function

$F(x,y)$ which is positively homogeneous of degree one in y^i and satisfies the usual conditions of H. Rund [120]. Let F_{n-1} be a hypersurface of F_n given by the equation $x^i := x^i(u^\alpha)$, where u^α are the *Gaussian coordinates* on $F_{n-1}(\alpha = 1,2,\cdots n-1)$. Suppose that the projection factor $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ is such that the rank of the matrix B^i_α is $(n-1)$. The element of support y^i of F_n is to be taken tangential to F_{n-1}, i.e.,

$$y^i = B^i_\alpha(u)v^\alpha. \tag{2.1}$$

Thus v^α is the element of support of F_{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}(u,\dot{u})$ of F_{n-1} is given by

$$g_{\alpha\beta}(u,\dot{u}) = g_{ij}(x,y)B^{ij}_{\alpha\beta}, \tag{2.2}$$

and the torsion tensor $C_{\alpha\beta\gamma}$ of F_{n-1} is given by

$$C_{\alpha\beta\gamma} = C_{ijk}B^{ijk}_{\alpha\beta\gamma} \tag{2.3}$$

where
$$B^{ij..k}_{\alpha\beta...\delta} = B^i_\alpha \cdots B^k_\delta .$$

At each point u^α of F_{n-1}, a unit normal vector $N^{*i}(u,v)$ is defined by

(2.4) a) $$g_{ij}(x(u),\ y(u,v))B^i_\alpha N^{*j} = 0,$$

b) $$g_{ij}N^{*i}N^{*j} = 1$$

and is called as the secondary normal to the hypersurface at the point. The inverse of B^i_α is B^α_i defined by

(2.5) a) $$B_i^{\alpha} = g^{\alpha\beta} B_{\beta}^{j} g_{ij}$$

so that.

b) $$B_{\alpha}^{i} B_{i}^{\beta} = \delta_{\alpha}^{\beta}.$$

Rund [120] defined $\overset{0}{\delta}$-operator by

(2.6) $$\overset{0}{\delta}_{\beta} X_{\alpha}^{i} = \partial_{\beta} X_{\alpha}^{i} + \Gamma_{jk}^{i} X_{\alpha}^{j} B_{\beta}^{k} - \Gamma_{\alpha\beta}^{\gamma} X_{\gamma}^{i},$$

where X_{α}^{i} is a mixed tensor.

In particular, we have

(2.7) $$I_{\alpha\beta}^{i} = \overset{0}{\delta}_{\beta} B_{\alpha}^{i} = B_{\alpha\beta}^{i} + \Gamma_{jk}^{*i} B_{\alpha\beta}^{jk}$$

and

(2.8) $$I_{\alpha\beta}^{i} = N^{*i} \Omega_{\alpha\beta}^{*}$$

where $$B_{\alpha\beta}^{i} = \frac{\partial^2 x^i}{\partial u^{\alpha} \partial u^{\beta}}.$$

For tensors T_j^i and T_{β}^{α} of F_n and F_{n-1} respectively, we have

(2.9) a) $$\overset{0}{\delta}_{\alpha} T_j^i = T_{j;k}^i B_{\alpha}^k,$$

b) $$\overset{0}{\delta}_{\alpha} T_{\beta}^{\gamma} = T_{\beta;\alpha}^{\gamma},$$

where $T_{j;k}^{i}$ denotes the δ-covariant differentiation [120]. The induced connection $IC\Gamma = (\Gamma_{\beta\gamma}^{*\alpha}, G_{\beta}^{\alpha}, C_{\beta\gamma}^{\alpha})$ of F_{n-1} induced by the Cartan's connection $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{ok}^{*i}, C_{jk}^{i})$ of

F_n is given by [50]

(2.10) a) $$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i}(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}) + M^{\alpha}_{\beta}H_{\gamma},$$

b) $$G^{\alpha}_{\beta} = B^{\alpha}_{i}(B^{i}_{o\beta} + \Gamma^{*i}_{oj}B^{j}_{\beta})$$

and

c) $$C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}.$$

Where

(2.11) a) $$M_{\beta\gamma} = N^{*}_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma},$$

b) $$H_{\beta} = N^{*}_{i}(B^{i}_{o\beta} + \Gamma^{*i}_{oj}B^{j}_{\beta}).$$

The quantities $M_{\beta\gamma}$ and H_{β} are called *second fundamental v-tensor* and *normal curvature vector* respectively [50].

The second fundamental h-tensor $H_{\beta\gamma}$ is defined as

(2.12) a) $$H_{\beta\gamma} = N^{*}_{i}(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}) + M_{\beta}H_{\gamma},$$

where

b) $$M_{\beta} = N^{*}_{i}C^{i}_{jk}B^{j}_{\beta}N^{*k}$$

and

c) $$\Omega^{*}_{\beta\gamma} = N^{*}_{i}(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}).$$

The relative h- and v-covariant derivative of projection factor B^i_α with respect to $IC\Gamma$ are given by

(2.13) a) $$B^i_{\alpha|\beta} = H_{\alpha\beta} N^{*i}$$

and

b) $$B^i_\alpha |_\beta = M_{\alpha\beta} N^{*i}.$$

From equations (2.11b) and (2.12), the equation (2.13a) may be written as

(2.14) $$B^i_{\alpha|\beta} = N^{*i}[\Omega^*_{\alpha\beta} + M_\alpha \Omega^*_{0\beta}].$$

3. Hypersurface of a C^δ-Recurrent Finsler Space.

A Finsler space F_n will be called a C^δ-recurrent Finsler space if there exist a non-zero vector λ_l such that

(3.1) $$C_{ijk;l} = \lambda_l C_{ijk}.$$

Taking δ-covariant derivative of (2.3) with respect to u^σ and using (2.7), we get

(3.2) $$C_{\alpha\beta\gamma;\sigma} = C_{ijk;l} B^{ijkl}_{\alpha\beta\gamma\sigma} + C_{ijk}[B^{ij}_{\alpha\beta} I^k_{\gamma\sigma} + B^{jk}_{\beta\gamma} I^i_{\alpha\sigma} + B^{ik}_{\alpha\gamma} I^j_{\beta\sigma}].$$

Using equations (3.1) and (2.3) in the equation (3.2), we get

(3.3) $$C_{\alpha\beta\gamma;\sigma} = \lambda_\sigma C_{\alpha\beta\gamma} + C_{ijk}[B^{ij}_{\alpha\beta} I^k_{\gamma\sigma} + B^{jk}_{\beta\gamma} I^i_{\alpha\sigma} + B^{ik}_{\alpha\gamma} I^j_{\beta\sigma}],$$

where

(3.4) $$\lambda_\sigma = \lambda_l B^l_\sigma.$$

Suppose that the hypersurface F_{n-1} *is totally geodesic* i.e. $\Omega^*_{\alpha\beta} = 0$. In view of $\Omega^*_{\alpha\beta} = 0$, the equation (2.8) implies $I^i_{\alpha\beta} = 0$. Using $I^i_{\alpha\beta} = 0$ in (3.3), we have

(3.5) $$C_{\alpha\beta\gamma,\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}.$$

Thus, we conclude

Theorem 3.1. *A totally geodesic hypersurface F_{n-1} of a C^δ-recurrent Finsler space F_n is itself C^δ-recurrent with the recurrence vector λ_σ given by* (3.4).

If λ_σ=0 in equation (3.4), recurrence vector λ_l is normal to the hypersurface F_{n-1}. A hypersurface is said to be C^δ*-symmetric* if the torsion tensor $C_{\alpha\beta\gamma}$ is covariant constant, i.e. $C_{\alpha\beta\gamma;\sigma}$= 0. Using the condition of C^δ-symmetric hypersurface in the equation (3.5), we get $\lambda_\sigma = 0$. Thus if a totally geodesic hypersurface F_{n-1} of a C^δ-recurrent Finsler space is C^δ-symmetric, the recurrence vector λ_l of F_n is normal to the hypersurface F_{n-1}.

Conversely, suppose that the recurrence vector λ_l of F_n is normal to the hypersurface F_{n-1}. This supposition reduces the equation (3.5) into

$$C_{\alpha\beta\gamma,\sigma} = 0$$

which shows that the hypersurface is C^δ-symmetric.

Thus, we have

Theorem 3.2. *A necessary and sufficient condition for a totally geodesic hypersurface F_{n-1} of a C^δ-recurrent Finsler space to be C^δ-symmetric is that the recurrence vector λ_l of F_n be normal to the hypersurface F_{n-1}.*

Now we try to find the condition under which an umbilical hypersurface is C^δ-recurrent. A hypersurface F_{n-1} is called *umbilical* if its lines of curvature are

indeterminate. The condition for this is

(3.6) $$\Omega^*_{\alpha\beta} = \rho g_{\alpha\beta},$$

where

$$\rho = \Omega^*_{\alpha\beta} g^{\alpha\beta} /(n-1) = \frac{M^*}{n-1}.$$

Let us assume that the hypersurface F_{n-1} of a C^δ-recurrent Finsler space is umbilical. Using equations (3.6) and (2.8) in the equation (3.3), we get

(3.7) $$C_{\alpha\beta\gamma;\sigma} = \lambda_\sigma C_{\alpha\beta\gamma} + \rho C_{ijk}[B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} + B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}].$$

Therefore

$$C_{\alpha\beta\gamma;\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}$$

if and only if

(3.8) $$C_{ijk}[B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} + B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}] = 0.$$

Thus, we have

Theorem 3.3. *An umbilical hypersurface F_{n-1} of a C^δ-recurrent Finsler space F_n whose recurrence vector is not normal to the hypersurface F_{n-1} is C^δ-recurrent if and only if* (3.8) *holds good.*

If the recurrence vector λ_l of F_n is normal to the hypersurface F_{n-1}. Then the equation (3.7) reduces to

$$C_{\alpha\beta\gamma;\sigma} = \rho C_{ijk}[B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} + B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}].$$

This equation shows that the hypersurface F_{n-1} is C^{δ}-symmetric if and only if (3.8) is satisfied. This leads to,

Theorem 3.4. *An umbilical hypersurface* F_{n-1} *of a* $\overset{\delta}{C}$ *-recurrent Finsler space* F_n *whose recurrence vector is normal to* F_{n-1} *is symmetric if and only if the relation* (3.8) *is satisfied.*

4. Hypersurface of a C^{δ}-Birecurrent Finsler Space

A Finsler space F_n will be called a C^{δ}-birecurrent Finsler space if there exists a non-zero tensor a_{ml} such that

(4.1) $$C_{ijk;m;l} = a_{ml} C_{ijk}\,.$$

Taking δ-covariant derivative of (2.3) with respect to u^{σ} and u^{ε} successively, we get

(4.2) $$C_{\alpha\beta\gamma;\sigma;\varepsilon} = C_{ijk;m;l} B^{ijkml}_{\alpha\beta\gamma\sigma\varepsilon} + C_{ijk;m}[B^{ijk}_{\alpha\beta\gamma} I^{m}_{\sigma\varepsilon}$$

$$+ B^{ijm}_{\alpha\beta\sigma} I^{k}_{\gamma\varepsilon} + B^{ikm}_{\alpha\gamma\sigma} I^{j}_{\beta\varepsilon} + B^{jkm}_{\beta\gamma\sigma} I^{i}_{\alpha\varepsilon}] + C_{ijk}[B^{ij}_{\alpha\beta} I^{k}_{\gamma\sigma}$$

$$+ B^{jk}_{\beta\gamma} I^{i}_{\alpha\sigma} + B^{ik}_{\alpha\gamma} I^{j}_{\beta\sigma}]_{;\varepsilon} + C_{ijk;l} B^{l}_{\varepsilon}[B^{ij}_{\alpha\beta} I^{k}_{\gamma\sigma} + B^{jk}_{\beta\gamma} I^{i}_{\alpha\sigma} + B^{ik}_{\alpha\gamma} I^{j}_{\beta\sigma}],$$

which, in view of (2.3) and (4.1), becomes

(4.3) $$C_{\alpha\beta\gamma;\sigma;\varepsilon} = a_{\sigma\varepsilon} C_{\alpha\beta\gamma} + C_{ijk;m}[B^{ijk}_{\alpha\beta\gamma} I^{m}_{\sigma\varepsilon} + B^{ijm}_{\alpha\beta\sigma} I^{k}_{\gamma\varepsilon}$$

$$+ B^{ikm}_{\alpha\gamma\sigma} I^{j}_{\beta\varepsilon} + B^{jkm}_{\beta\gamma\sigma} I^{i}_{\alpha\varepsilon}] + C_{ijk}[B^{ij}_{\alpha\beta} I^{k}_{\gamma\sigma} + B^{jk}_{\beta\gamma} I^{i}_{\alpha\sigma} + B^{ik}_{\alpha\gamma} I^{j}_{\beta\sigma}]_{;\varepsilon}$$

$$+ C_{ijk;l} B^{l}_{\varepsilon}[B^{ij}_{\alpha\beta} I^{k}_{\gamma\sigma} + B^{jk}_{\beta\gamma} I^{i}_{\alpha\sigma} + B^{ik}_{\alpha\gamma} I^{j}_{\beta\sigma}],$$

where

(4.4) $$a_{\sigma\varepsilon} = a_{ml} B^{ml}_{\sigma\varepsilon}.$$

Suppose that the hypersurface is totally geodesic i.e. $\Omega^*_{\alpha\beta} = 0$. Then, in view of (2.8), $I^{\iota}_{\alpha\beta} = 0$ and hence the equation (4.3) implies

(4.5) $$C_{\alpha\beta\gamma.\sigma;\varepsilon} = a_{\sigma\varepsilon} C_{\alpha\beta\gamma}.$$

Thus, we conclude

Theorem 4.1. *A totally geodesic hypersurface F_{n-1} of a C^{δ}-birecurrent Finsler space F_n is C^{δ}-birecurrent with recurrence tensor $a_{\sigma\varepsilon}$ given by* (4.4).

A hypersurface F_{n-1} is said to be C^{δ}*-bisymmetric* if the torsion tensor $C_{\alpha\beta\gamma}$ satisfies $C_{\alpha\beta\gamma;\sigma;\varepsilon} = 0$. From (4.5), it is obvious that the hypersurface F_{n-1} is C^{δ}-bisymmetric if and only if $a_{\sigma\varepsilon} = 0$. This leads to:

Theorem 4.2. *A necessary and sufficient condition for a totally geodesic hypersurface F_{n-1} of a C^{δ}-birecurrent Finsler space F_n to be C^{δ}-bisymmetric is that the recurrence tensor $a_{\sigma\varepsilon} = 0$.*

Let us assume that the hypersurface F_{n-1} of a C^{δ}-birecurrent Finsler space F_n is umbilical. Then, in view of (3.6) and (2.8), the equation (4.3) gives

(4.6) $$\begin{aligned} C_{\alpha\beta\gamma;\sigma;\varepsilon} &= a_{\sigma\varepsilon} C_{\alpha\beta\gamma} + \rho C_{ijk;m} [B^{ijk}_{\alpha\beta\gamma} N^{*m} g_{\sigma\varepsilon} \\ &+ B^{ijm}_{\alpha\beta\sigma} N^{*k} g_{\gamma\varepsilon} + B^{jkm}_{\beta\gamma\sigma} N^{*i} g_{\alpha\varepsilon} + B^{ikm}_{\alpha\gamma\sigma} N^{*j} g_{\beta\varepsilon}] \\ &+ C_{ijk} [\rho B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + \rho B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} + \rho B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}]_{;\varepsilon} \\ &+ C_{ijk;l} B^{l}_{\varepsilon} [\rho B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + \rho B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} + \rho B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}]. \end{aligned}$$

Therefore

$$C_{\alpha\beta\gamma;\sigma;\varepsilon} = a_{\sigma\varepsilon} C_{\alpha\beta\gamma} \tag{4.7}$$

if and only if

$$\begin{aligned} &\rho C_{ijk;m}[B^{ijk}_{\alpha\beta\gamma} N^{*m} g_{\sigma\varepsilon} + B^{ijm}_{\alpha\beta\sigma} N^{*k} g_{\gamma\varepsilon} + B^{jkm}_{\beta\gamma\sigma} N^{*i} g_{\alpha\varepsilon} \\ &+ B^{ikm}_{\alpha\gamma\sigma} N^{*j} g_{\beta\varepsilon}] + C_{ijk}[\rho B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + \rho B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} \\ &+ \rho B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}]_{;\varepsilon} + \rho C_{ijk;l} B^{l}_{\varepsilon}[B^{ij}_{\alpha\beta} N^{*k} g_{\gamma\sigma} + B^{jk}_{\beta\gamma} N^{*i} g_{\alpha\sigma} \\ &+ B^{ik}_{\alpha\gamma} N^{*j} g_{\beta\sigma}] = 0. \end{aligned} \tag{4.8}$$

Thus, we have

Theorem 4.3. *An umbilical hypersurface F_{n-1} of a C^{δ}-birecurrent Finsler space F_n is C^{δ}-birecurrent if and only if* (4.8) *holds good.*

Let us assume that the recurrence tensor $a_{\sigma\varepsilon} = 0$. Then from (4.7) we may conclude

Theorem 4.4. *In case of umbilical hypersurface F_{n-1} of a C^{δ}-birecurrent Finsler space F_n the torsion tensor $C_{\alpha\beta\gamma}$ satisfies the condition of C^{δ}-bisymmetric hypersurface i.e., $C_{\alpha\beta\gamma;\sigma;\varepsilon} = 0$ if and only if* (4.8) *and $a_{\sigma\varepsilon} = 0$ hold.*

5. Hypersurface of a C^h-Recurrent Finsler Space

A Finsler space whose torsion tensor C_{ijk} satisfies the recurrence property with respect to Cartan connection Γ^{*i}_{jk} was discussed by Makoto Matsumoto [39] and called by him as C^h-recurrent space. Thus a C^h-recurrent space is characterized by the condition

$$C_{ijk|m} = \lambda_m C_{ijk}, \quad C_{ijk} \neq 0 \tag{5.1}$$

The non-zero covariant vector field λ_m is the recurrence vector field. In this section we study the properties of hypersurface of such space.

Differentiating (2.3) covariantly with respect to u^σ in the sense of Cartan and using (2.14), we get

(5.2) $$C_{\alpha\beta\gamma|\sigma} = C_{ijk|m} B^{ijkm}_{\alpha\beta\gamma\sigma} + C_{ijk}[B^{ij}_{\alpha\beta} N^{*k}(\Omega^*_{\gamma\sigma} + M_\gamma \Omega^*_{0\sigma})$$

$$+ B^{jk}_{\beta\gamma} N^{*i}(\Omega^*_{\alpha\sigma} + M_\alpha \Omega^*_{0\sigma}) + B^{ik}_{\alpha\gamma} N^{*j}(\Omega^*_{\beta\sigma} + M_\beta \Omega^*_{0\sigma})].$$

Using (5.1) and (2.3) in the equation (5.2), we find

(5.3) $$C_{\alpha\beta\gamma|\sigma} = \lambda_\sigma C_{\alpha\beta\gamma} + C_{ijk}[B^{ij}_{\alpha\beta} N^{*k}(\Omega^*_{\gamma\sigma} + M_\gamma \Omega^*_{0\sigma})$$

$$+ B^{jk}_{\beta\gamma} N^{*i}(\Omega^*_{\alpha\sigma} + M_\alpha \Omega^*_{0\sigma}) + B^{ik}_{\alpha\gamma} N^{*j}(\Omega^*_{\beta\sigma} + M_\beta \Omega^*_{0\sigma})],$$

where

(5.4) $$\lambda_\sigma = \lambda_m B^m_\sigma.$$

Suppose that the hypersurface F_{n-1} is totally geodesic. Using the condition of totally geodesic hypersurface in the equation (5.3), we get

(5.5) $$C_{\alpha\beta\gamma|\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}.$$

Therefore, we have

Theorem 5.1. *A totally geodesic hypersurface F_{n-1} of a C^h-recurrent Finsler space F_n is itself C^h-recurrent with recurrence vector λ_σ given by* (5.4) .

Suppose that the hypersurface F_{n-1} is C^h-*symmetric* i.e. $C_{\alpha\beta\gamma|\sigma} = 0$. Using $C_{\alpha\beta\gamma|\sigma} = 0$ in equation (5.5), we get $\lambda_\sigma = 0$ i.e., the recurrence vector λ_m of Finsler

space F_n is normal to the hypersurface F_{n-1}. Conversely, suppose that the recurrence vector of Finsler space F_n is normal to the hypersurface F_{n-1}. Then from (5.5), we get

$$C_{\alpha\beta\gamma|\sigma} = 0 .$$

Thus, we see that the hypersurface is C^h-symmetric. From the above discussion, we conclude

Theorem 5.2. *A necessary and sufficient condition for a totally geodesic hypersurface F_{n-1} of a C^h-recurrent Finsler space F_n to be C^h-symmetric is that the recurrence vector λ_m of F_n be normal to the hypersurface F_{n-1}.*

Let us assume that the hypersurface F_{n-1} of a C^h-recurrent Finsler space F_n to be umbilical. Using (3.6) in (5.3), we get

$$C_{\alpha\beta\gamma|\sigma} = \lambda_\sigma C_{\alpha\beta\gamma} + \rho C_{ijk}[B^{ij}_{\alpha\beta}N^{*k}(g_{\gamma\sigma} + v_\sigma M_\gamma) \tag{5.6}$$

$$+ B^{jk}_{\beta\gamma}N^{*i}(g_{\alpha\sigma} + v_\sigma M_\alpha) + B^{ik}_{\alpha\gamma}N^{*j}(g_{\beta\sigma} + v_\sigma M_\beta)] .$$

Therefore

$$C_{\alpha\beta\gamma|\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}$$

if and only if

$$C_{ijk}[B^{ij}_{\alpha\beta}N^{*k}(g_{\gamma\sigma} + v_\sigma M_\gamma) + B^{jk}_{\beta\gamma}N^{*i}(g_{\alpha\sigma} + v_\sigma M_\alpha) \tag{5.7}$$

$$+ B^{ik}_{\alpha\gamma}N^{*j}(g_{\beta\sigma} + v_\sigma M_\beta)] = 0 .$$

Thus, we have

Theorem 5.3. *If the hypersurface F_{n-1} of a C^h-recurrent Finsler space F_n whose recurrence vector is not normal to the hypersurface F_{n-1} is umbilical then the hypersurface F_{n-1} is C^h-recurrent if and only if* (5.7) *holds good.*

Let us consider that the recurrence vector of F_n is normal to the hypersurface F_{n-1}. Then from (5.6), we conclude

Theorem 5.4. *For an umbilical hypersurface F_{n-1} of a C^h-recurrent Finsler space F_n the tensor $C_{\alpha\beta\gamma}$ is a covariant constant if and only if* (5.7) *and $\lambda_\sigma = 0$ hold.*

6. Hypersurface of a C^h-Birecurrent Finsler Space

A Finsler space whose torsion tensor C_{ijk} satisfies the birecurrence property with respect to Cartan connection Γ^{*i}_{jk} was discussed by P. N. Pandey and Reema Verma [113], called by them as C^h-birecurrent space. Thus, a C^h-birecurrent space is characterized by the condition

$$C_{ijk|m|l} = a_{ml} C_{ijk}, \quad C_{ijk} \neq 0. \tag{6.1}$$

The non-zero tensor field a_{ml} is the recurrence tensor field. In this section we study the properties of the hypersurface of such space.

Differentiating (6.1) covariantly with respect to u^σ and u^ε successively in the sense of Cartan and using (2.14), we get

$$\begin{aligned} C_{\alpha\beta\gamma|\sigma|\varepsilon} &= C_{ijk|m|l} B^{ijkml}_{\alpha\beta\gamma\sigma\varepsilon} + C_{ijk|m}[B^{ijk}_{\alpha\beta\gamma} N^{*m}(\Omega^*_{\sigma\varepsilon} + M_\sigma \Omega^*_{0\varepsilon}) \\ &+ B^{ijm}_{\alpha\beta\sigma} N^{*k}(\Omega^*_{\gamma\varepsilon} + M_\gamma \Omega^*_{0\varepsilon}) + B^{ikm}_{\alpha\gamma\sigma} N^{*j}(\Omega^*_{\beta\varepsilon} + M_\beta \Omega^*_{0\varepsilon}) \\ &+ B^{jkm}_{\beta\gamma\sigma} N^{*i}(\Omega^*_{\alpha\varepsilon} + M_\alpha \Omega^*_{o\varepsilon})] + C_{ijk}[B^{ij}_{\alpha\beta} N^{*k}(\Omega^*_{\gamma\sigma} + \Omega^*_{0\sigma} M_\gamma) \end{aligned} \tag{6.2}$$

$$+B^{jk}_{\beta\gamma}N^{*i}(\Omega^{*}_{\alpha\sigma}+M_{\alpha}\Omega^{*}_{0\sigma})+B^{ik}_{\alpha\gamma}N^{*j}(\Omega^{*}_{\beta\sigma}+M_{\beta}\Omega^{*}_{0\sigma})]_{|\varepsilon}$$

$$+C_{ijkl|l}B^{l}_{\varepsilon}[B^{ij}_{\alpha\beta}N^{*k}(\Omega^{*}_{\gamma\sigma}+M_{\gamma}\Omega^{*}_{0\sigma})+B^{jk}_{\beta\gamma}N^{*i}(\Omega^{*}_{\alpha\sigma}+M_{\alpha}\Omega^{*}_{0\sigma})$$

$$+B^{ik}_{\alpha\gamma}N^{*j}(\Omega^{*}_{\beta\sigma}+M_{\beta}\Omega^{*}_{0\sigma})].$$

Using equations (6.1) and (2.3) in the equation (6.2), we get

(6.3)
$$C_{\alpha\beta\gamma|\sigma|\varepsilon}=a_{\sigma\varepsilon}C_{\alpha\beta\gamma}+C_{ijk|m}[B^{ijk}_{\alpha\beta\gamma}N^{*m}(\Omega^{*}_{\sigma\varepsilon}+M_{\sigma}\Omega^{*}_{0\varepsilon})$$

$$+B^{ijm}_{\alpha\beta\sigma}N^{*k}(\Omega^{*}_{\gamma\varepsilon}+M_{\gamma}\Omega^{*}_{0\varepsilon})+B^{ikm}_{\alpha\gamma\sigma}N^{*j}(\Omega^{*}_{\beta\varepsilon}+M_{\beta}\Omega^{*}_{0\varepsilon})$$

$$+B^{jkm}_{\beta\gamma\sigma}N^{*i}(\Omega^{*}_{\alpha\varepsilon}+M_{\alpha}\Omega^{*}_{0\varepsilon})]+C_{ijk}[B^{ij}_{\alpha\beta}N^{*k}(\Omega^{*}_{\gamma\sigma}+M_{\gamma}\Omega^{*}_{0\sigma})$$

$$+B^{jk}_{\beta\gamma}N^{*i}(\Omega^{*}_{\alpha\sigma}+M_{\alpha}\Omega^{*}_{0\sigma})+B^{ik}_{\alpha\gamma}N^{*j}(\Omega^{*}_{\beta\sigma}+M_{\beta}\Omega^{*}_{0\sigma})]_{|\varepsilon}$$

$$+C_{ijk|l}B^{l}_{\varepsilon}[B^{ij}_{\alpha\beta}N^{*k}(\Omega^{*}_{\gamma\sigma}+M_{\gamma}\Omega^{*}_{0\sigma})+B^{jk}_{\beta\gamma}N^{*i}(\Omega^{*}_{\alpha\sigma}+M_{\alpha}\Omega^{*}_{0\sigma})$$

$$+B^{ik}_{\alpha\gamma}N^{*j}(\Omega^{*}_{\beta\sigma}+M_{\beta}\Omega^{*}_{0\sigma})],$$

where

(6.4)
$$a_{\sigma\varepsilon}=a_{ml}B^{ml}_{\sigma\varepsilon}.$$

Suppose that the hypersurface is totally geodesic i.e., $\Omega^{*}_{\alpha\beta}=0$. Using this condition in equation (6.3), we get

(6.5)
$$C_{\alpha\beta\gamma|\sigma|\varepsilon}=a_{\sigma\varepsilon}C_{\alpha\beta\gamma}.$$

Thus, we conclude

Theorem 6.1. *A totally geodesic hypersurface* F_{n-1} *of a* C^h*-birecurrent Finsler space* F_n *is* C^h*-birecurrent.*

Suppose that the hypersurface is C^h*-bisymmetric* i.e., the torsion tensor $C_{\alpha\beta\gamma}$ satisfy $C_{\alpha\beta\gamma|\sigma|\varepsilon} = 0$. Using this condition in the equation (6.5) we get $a_{\sigma\varepsilon} = 0$. Conversely suppose that $a_{\sigma\varepsilon} = 0$. Then from (6.5) we get

$$C_{\alpha\beta\gamma|\sigma|\varepsilon} = 0,$$

which is the characterizing condition of C^h-bisymmetric hypersurface. Thus, we conclude

Theorem 6.2. *A necessary and sufficient condition for a totally geodesic hypersurface* F_{n-1} *of a* C^h*-birecurrent Finsler space* F_n *to be* C^h*-bisymmetric is that* $a_{\sigma\varepsilon} = 0$.

Let us assume that the hypersurface F_{n-1} of a C^h-birecurrent space F_n to be umbilical. Using equation (3.6) in the equation (6.3), we get

$$(6.6) \qquad C_{\alpha\beta\gamma|\sigma|\varepsilon} = a_{\sigma\varepsilon} C_{\alpha\beta\gamma} + \rho C_{ijk|m} [B^{ijk}_{\alpha\beta\gamma} N^{*m} (g_{\sigma\varepsilon} + v_\varepsilon M_\sigma)$$

$$+ B^{ijm}_{\alpha\beta\sigma} N^{*k} (g_{\gamma\varepsilon} + v_\varepsilon M_\gamma) + B^{ikm}_{\alpha\gamma\sigma} N^{*j} (g_{\beta\varepsilon} + v_\varepsilon M_\beta)$$

$$+ B^{jkm}_{\beta\gamma\sigma} N^{*i} (g_{\alpha\varepsilon} + v_\varepsilon M_\alpha)] + C_{ijk} [\rho B^{ij}_{\alpha\beta} N^{*k} (g_{\gamma\sigma} + v_\sigma M_\gamma)$$

$$+ \rho B^{jk}_{\beta\gamma} N^{*i} (g_{\alpha\sigma} + v_\sigma M_\alpha) + \rho B^{ik}_{\alpha\gamma} N^{*j} (g_{\beta\sigma} + v_\sigma M_\beta)]_{|\varepsilon}$$

$$+ \rho C_{ijk|l} B^l_\varepsilon [B^{ij}_{\alpha\beta} N^{*k} (g_{\gamma\sigma} + v_\sigma M_\gamma) + B^{jk}_{\beta\gamma} N^{*i} (g_{\alpha\sigma} + M_\alpha v_\sigma)$$

$$+ B^{ik}_{\alpha\gamma} N^{*j} (g_{\beta\sigma} + v_\sigma M_\beta)].$$

Therefore

$$C_{\alpha\beta\gamma|\sigma|\varepsilon} = a_{\sigma\varepsilon} C_{\alpha\beta\gamma}$$

if and only if

(6.7) $$\rho C_{ijk|m}[B^{ijk}_{\alpha\beta\gamma} N^{*m}(g_{\sigma\varepsilon} + v_\varepsilon M_\sigma) + B^{ijm}_{\alpha\beta\sigma} N^{*k}(g_{\gamma\varepsilon} + v_\varepsilon M_\gamma)$$

$$+ B^{ikm}_{\alpha\gamma\sigma} N^{*j}(g_{\beta\varepsilon} + v_\varepsilon M_\beta) + B^{jkm}_{\beta\gamma\sigma} N^{*i}(g_{\alpha\varepsilon} + v_\varepsilon M_\alpha)]$$

$$+ C_{ijk}[\rho B^{ij}_{\alpha\beta} N^{*k}(g_{\gamma\sigma} + v_\sigma M_\gamma) + \rho B^{jk}_{\beta\gamma} N^{*i}(g_{\alpha\sigma} + M_\alpha v_\sigma)$$

$$+ \rho B^{ik}_{\alpha\gamma} N^{*j}(g_{\beta\sigma} + v_\sigma M_\beta)]_{|\varepsilon} + \rho C_{ijk|l} B^l_\varepsilon [B^{ij}_{\alpha\beta} N^{*k}$$

$$(g_{\gamma\sigma} + M_\gamma v_\sigma) + B^{jk}_{\beta\gamma} N^{*i}(g_{\alpha\sigma} + v_\sigma M_\alpha) + B^{ik}_{\alpha\gamma} N^{*j}(g_{\beta\sigma} + v_\sigma M_\beta)] = 0.$$

Thus, we have

Theorem 6.3. *If the hypersurface F_{n-1} of a C^h-birecurrent Finsler space F_n is umbilical then the hypersurface F_{n-1} is C^h-birecurrent if and only if* (6.7) *holds good.*

Let us assume that the recurrence tensor $a_{\sigma\varepsilon} = 0$. Then from (6.6), we conclude

Theorem 6.4. *In case of umbilical hypersurface F_{n-1} of a C^h-birecurrent Finsler space F_n the tensor $C_{\alpha\beta\gamma}$ satisfies the characterizing condition of a C^h-bisymmetric hypersurface if and only if* (6.7) and $a_{\sigma\varepsilon} = 0$ hold.

7. Hypersurface of a C2-Like Finsler Space

Definition 7.1.

A non-Riemannian Finsler space $F_n (n \geq 2)$ with $C^2 \neq 0$ is called C2-like if the $h(hv)$-torsion tensor C_{ijk} is written in the form

(7.1) $$C_{ijk} = C_i C_j C_k / C^2.$$

Using equation (7.1) in (2.3), we get

(7.2) $$C_{\alpha\beta\gamma}=\frac{C_i C_j C_k}{C^2}B^i_\alpha B^j_\beta B^k_\gamma .$$

Equation (7.2) may be written as

(7.3) $$C_{\alpha\beta\gamma}=\frac{C_\alpha C_\beta C_\gamma}{C^2}.$$

Thus, we have

Theorem 7.1. *A hypersurface of a C2-like Finsler space is a C2-like space.*

The difference between the intrinsic and induced connection parameters of a hypersurface has been obtained by Rund [121], which is as follows:

(7.4) $$'\Gamma^*_{\alpha\beta\gamma}-\Gamma^*_{\alpha\beta\gamma}=N^{*j}C_{hkj}[(B^{hk}_{\beta\gamma}\Omega^*_{\alpha\varepsilon}+B^{hk}_{\alpha\beta}\Omega^*_{\gamma\varepsilon}-B^{hk}_{\gamma\alpha}\Omega^*_{\beta\varepsilon})\dot{u}^\varepsilon$$
$$-(C^\delta_{\beta\gamma}B^{hk}_{\delta\alpha}+C^\delta_{\alpha\beta}B^{hk}_{\delta\gamma}-C^\delta_{\gamma\alpha}B^{hk}_{\delta\beta})\Omega^*_{\varepsilon\lambda}\dot{u}^\varepsilon\dot{u}^\lambda].$$

Using the characterizing condition of a C2-like Finsler space in (7.4), we get

(7.5) $$\Lambda_{\alpha\beta\gamma}=\frac{N^{*j}C_h C_k C_j}{C^2}[(B^{hk}_{\beta\gamma}\Omega^*_{\alpha\varepsilon}+B^{hk}_{\alpha\beta}\Omega^*_{\gamma\varepsilon}-B^{hk}_{\gamma\alpha}\Omega^*_{\beta\varepsilon})$$
$$\dot{u}^\varepsilon-(C^\delta_{\beta\gamma}B^{hk}_{\delta\alpha}+C^\delta_{\alpha\beta}B^{hk}_{\delta\gamma}-C^\delta_{\gamma\alpha}B^{hk}_{\delta\beta})\Omega^*_{\varepsilon\lambda}\dot{u}^\varepsilon\dot{u}^\lambda]$$

Equation (7.5) may be written as

(7.6) $$\Lambda_{\alpha\beta\gamma}=\rho[\frac{1}{C^2}(C_\beta C_\gamma\Omega^*_{\alpha 0}+C_\alpha C_\beta\Omega^*_{\gamma 0}-C_\alpha C_\gamma\Omega^*_{\beta 0})-C_{\alpha\beta\gamma}\Omega^*_{00}]$$

where $\rho=C_i N^i$.

Thus, we have

Using equation (7.1) in (2.3), we get

$$C_{\alpha\beta\gamma} = \frac{C_i C_j C_k}{C^2} B^i_\alpha B^j_\beta B^k_\gamma . \tag{7.2}$$

Equation (7.2) may be written as

$$C_{\alpha\beta\gamma} = \frac{C_\alpha C_\beta C_\gamma}{C^2} . \tag{7.3}$$

Thus, we have

Theorem 7.1. *A hypersurface of a C2-like Finsler space is a C2-like space.*

The difference between the intrinsic and induced connection parameters of a hypersurface has been obtained by Rund [121], which is as follows:

$$\begin{aligned} {}'\Gamma^*_{\alpha\beta\gamma} - \Gamma^*_{\alpha\beta\gamma} = N^{*j} C_{hkj}[(B^{hk}_{\beta\gamma}\Omega^*_{\alpha\varepsilon} + B^{hk}_{\alpha\beta}\Omega^*_{\gamma\varepsilon} - B^{hk}_{\gamma\alpha}\Omega^*_{\beta\varepsilon})\dot{u}^\varepsilon \\ -(C^\delta_{\beta\gamma} B^{hk}_{\delta\alpha} + C^\delta_{\alpha\beta} B^{hk}_{\delta\gamma} - C^\delta_{\gamma\alpha} B^{hk}_{\delta\beta})\Omega^*_{\varepsilon\lambda}\dot{u}^\varepsilon \dot{u}^\lambda] . \end{aligned} \tag{7.4}$$

Using the characterizing condition of a C2-like Finsler space in (7.4), we get

$$\begin{aligned} \Lambda_{\alpha\beta\gamma} = \frac{N^{*j} C_h C_k C_j}{C^2}[(B^{hk}_{\beta\gamma}\Omega^*_{\alpha\varepsilon} + B^{hk}_{\alpha\beta}\Omega^*_{\gamma\varepsilon} - B^{hk}_{\gamma\alpha}\Omega^*_{\beta\varepsilon}) \\ \dot{u}^\varepsilon - (C^\delta_{\beta\gamma} B^{hk}_{\delta\alpha} + C^\delta_{\alpha\beta} B^{hk}_{\delta\gamma} - C^\delta_{\gamma\alpha} B^{hk}_{\delta\beta})\Omega^*_{\varepsilon\lambda}\dot{u}^\varepsilon \dot{u}^\lambda] \end{aligned} \tag{7.5}$$

Equation (7.5) may be written as

$$\Lambda_{\alpha\beta\gamma} = \rho[\frac{1}{C^2}(C_\beta C_\gamma \Omega^*_{\alpha 0} + C_\alpha C_\beta \Omega^*_{\gamma 0} - C_\alpha C_\gamma \Omega^*_{\beta 0}) - C_{\alpha\beta\gamma}\Omega^*_{00}] \tag{7.6}$$

where $\rho = C_i N^i$.

Thus, we have

Theorem 7.2. *The necessary and sufficient condition that intrinsic and induced connection parameters of a hypersurface of a C2-like Finsler space be equal is either* $\Omega^*_{\alpha 0} = 0$ *or the vector* C_i *is tangential to the hypersurface* F_{n-1}.

The induced covariant differentiation of $C_\alpha = B^i_\alpha C_i$ is defined as follows [50]

$$C_{\alpha|\beta} = C_{i|h} B^i_\alpha B^h_\beta + \frac{\partial C_i}{\partial \dot{u}^\alpha} \Omega^*_{\beta 0} N^{*i} + \rho \Omega^*_{\alpha\beta} \tag{7.6}$$

Transvecting (7.6) by $\dot{u}^\beta$ we get

$$C_{\alpha|0} = C_{i|0} B^i_\alpha + \frac{\partial C_i}{\partial \dot{u}^\alpha} \Omega^*_{00} N^{*i} + \rho \Omega^*_{\alpha 0}. \tag{7.7}$$

Let us assume that the intrinsic and induced connection parameters are identical then by Theorem 7.2, either $\Omega^*_{\alpha 0} = 0$ or $\rho = 0$. If $\Omega^*_{\alpha 0} = 0$, then the equation (7.7) gives

$$C_{\alpha|0} = C_{i|0} B^i_\alpha. \tag{7.8}$$

If $\rho = 0$, then equation (7.1) shows that the tensor $M_{\alpha\beta}$ defined by

$$M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^{*k} \tag{7.9}$$

vanishes. C. M. Brown [13] discussed the properties of the hypersurface for this case. He showed that

$$\frac{\partial N^{*i}}{\partial \dot{u}^\alpha} = -M_\alpha N^{*i}, \text{ where } M_\alpha = C_{ijk} B^i_\alpha N^{*j} N^{*k}.$$

This relation and the condition $\rho = C_i N^i = 0$ imply

$$\frac{\partial C_i}{\partial \dot{u}^\alpha} N^{*i} = -C_i \frac{\partial N^{*i}}{\partial \dot{u}^\alpha} = -(C_i N^{*i}) M_\alpha = 0. \tag{7.10}$$

Equation (7.10) shows that the condition $\rho = 0$ again reduces the equation (7.7) to (7.8) in which the covariant differentiation on the left hand side of the equality is intrinsic as well as induced. Since a C2-like Finsler space is a Landsberg space if and only if $y^h C_{i|h} = 0$ or $C_{i|0} = 0$. Thus we have

Theorem 7.3. *If the induced and intrinsic connection parameters of a hypersurface of a C2-like Landsberg space are identical, then the hypersurface is Landsberg.*

The two normal curvature tensors, denoted by $I^i_{\alpha\beta}$ and $H^i_{\alpha\beta}$, are given by Rund [120] and Davies [20]. These are related by [120]

$$H^i_{\alpha\beta} = I^i_{\alpha\beta} + N^{*i} N^*_j C^j_{hk} B^h_\beta H^k_{\alpha\lambda} \dot{u}^\lambda \tag{7.11}$$

The above equation shows that

$$H^i_{\alpha\beta} \dot{u}^\beta = I^i_{\alpha\beta} \dot{u}^\beta = \Omega^*_{\alpha 0} N^{*i}. \tag{7.12}$$

From equation (7.1), (7.11) and (7.12), we get

$$H^i_{\alpha\beta} = I^i_{\alpha\beta} \frac{+\rho^2}{C^2} \Omega^*_{\alpha 0} C_\beta N^{*i}.$$

The condition $C_\beta = 0$ implies $C_i = 0$, which shows that the spaces F_n and F_{n-1} are Riemannian, which is a contradiction. Hence, we have the following

Theorem 7.4. *The necessary and sufficient condition that Rund's and Davies', normal curvature tensors of the hypersurface of a C2-like Finsler space are identical is that either $\Omega^*_{\alpha 0} = 0$ or $\rho = 0$.*

From Theorems 7.2, 7.3, and 7.4, we may conclude

Theorem 7.5. *If Rund's and Davies' normal curvature tensors of the hypersurface of a C2-like Landsberg space are equal then the induced and intrinsic connection parameters of the hypersurface are equal.*

Theorem 7.6. *If Rund's and Davies' normal curvature tensors of a hypersurface of a C2-like Landsberg space are equal then the hypersurface is also a Landsberg space.*

Next we try to find condition under which a hypersurface of a Finsler space satisfying T-condition also satisfies T-condition. The tensor T_{hijk} is defined as [34, 44].

$$T_{hijk} = C_{hij}|_k + C_{hij} l_k + C_{hik} l_j + C_{hjk} l_i + C_{ijk} l_h . \tag{7.13}$$

Taking v-covariant derivative of (2.3), we get

$$C_{\alpha\beta\gamma}|_\delta = C_{hij}|_k \, B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta + C_{hij} Z^h_{\alpha\delta} B^i_\beta B^j_\gamma \tag{7.14}$$

$$+ C_{hij} B^h_\alpha Z^i_{\beta\delta} B^j_\gamma + C_{hij} B^h_\alpha B^i_\beta Z^j_{\gamma\delta} ,$$

where

$$Z^h_{\alpha\delta} = B^h_\alpha |_\delta . \tag{7.15}$$

Using (2.11a), (2.13b), and (7.1) in the equation (7.15), we get

$$Z^h_{\alpha\delta} = \frac{\rho}{C^2} N^{*h} C_\alpha C_\delta . \tag{7.16}$$

In view of the equation (7.1) and (7.16), the equation (7.14) may be written as

$$C_{\alpha\beta\gamma}|_\delta = C_{hij}|_k \, B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta + \frac{\rho^2}{C^2}(3C_{\alpha\beta\gamma} C_\delta) . \tag{7.17}$$

From equation (7.17) and

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma}|_\delta + C_{\beta\gamma\delta} l_\alpha + C_{\alpha\gamma\delta} l_\beta + C_{\alpha\beta\delta} l_\gamma + C_{\alpha\beta\gamma} l_\delta ,$$

we get

$$T_{\alpha\beta\gamma\delta} = T_{hijk} B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta + \frac{3\rho^2}{C^2} C_{\alpha\beta\gamma} C_\delta . \tag{7.18}$$

The space F_n is said to satisfy T-condition if and only if $T_{hijk} = 0$. From this condition and equation (7.18), we conclude that the hypersurface of a C2-like Finsler space satisfying T-condition also satisfies T-condition if and only if $\rho = 0$ because the condition $C_{\alpha\beta\gamma} = 0$ implies that F_{n-1} is Riemannian. Thus, we have the following

Theorem 7.7. *The necessary and sufficient condition that the hypersurface F_{n-1} of a C2-like Finsler space F_n satisfying T-condition, satisfies T-condition is that the vector C_i is tangential to the hypersurface F_{n-1}.*

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Chapter V

HYPERSURFACE OF A RECURRENT FINSLER SPACE

1. Introduction

The recurrence of different curvature tensors due to different connections of L. Berwald and E. Cartan have been discussed by R. N. Sen [123], R. S. Mishra and H. D. Pande [65], R. B. Misra [58], P. N. Pandey and R. B. Misra [112], R. B. Misra and F. M. Meher [63], B. B. Sinha and S. P. Singh [127], P. N. Pandey [83, 84, 87-89, 94, 95, 97, 99-104] and others.

P. N. Pandey [89] and Shalini Dikshit [21] have discussed a Finsler space having recurrent Berwald curvature tensor. P. N. Pandey [89] established an important result concerning the recurrence vector of a recurrent Finsler space. He proved that the recurrence vector of a recurrent Finsler space is independent of directional arguments. Shalini Dikshit obtained a necessary and sufficient condition for the recurrence of associate Berwald curvature tensor of a Finsler space. U. P. Singh and G. C. Chaubey [124] studied the hypersurface of a Finsler space whose associate Berwald curvature tensor is recurrent, and called it a recurrent Finsler space. Thus, their recurrent Finsler space is characterized by

$$\mathcal{B}_m H_{hlkj} = a_m H_{hlkj}.$$

Since the characterizing condition of a recurrent Finsler space neither implies this condition nor is implied by this condition. In fact, they considered an affinely connected recurrent Finsler space in which both the conditions hold.

The aim of this chapter is to study the hypersurface of a recurrent Finsler space equipped with Berwald connection and to generalize the results of Singh and Chaubey.

2. Induced Berwald Connection

The induced and intrinsic Cartan connection coefficients $\Gamma^{*\alpha}_{\beta\gamma}$ and $'\Gamma^{*\alpha}_{\beta\gamma}$ [121] are related by

$$'\Gamma^{*\alpha}_{\beta\gamma} = \Lambda^{\alpha}_{\beta\gamma} + \Gamma^{*\alpha}_{\beta\gamma}, \tag{2.1}$$

where

$$g_{\varepsilon\gamma}\Lambda^{\varepsilon}_{\alpha\beta} = \Lambda_{\alpha\gamma\beta} = (M_{\beta\gamma}\Omega^{*}_{\alpha\sigma} + M_{\alpha\gamma}\Omega^{*}_{\beta\sigma} - M_{\alpha\beta}\Omega^{*}_{\gamma\sigma})\dot{u}^{\sigma}$$
$$- (M_{\lambda\alpha}C^{\lambda}_{\beta\gamma} + M_{\lambda\beta}C^{\lambda}_{\alpha\gamma} - M_{\lambda\gamma}C^{\lambda}_{\beta\alpha})\Omega^{*}_{\sigma\mu}\dot{u}^{\sigma}\dot{u}^{\mu}. \tag{2.2}$$

The normal curvature of the hypersurface F_{n-1} of a Finsler space F_n in the direction of $\dot{u}^{\sigma}$ is given by

$$k_n(u,\dot{u}) = (\Omega^{*}_{\sigma\lambda}\dot{u}^{\sigma}\dot{u}^{\lambda})F^{-2}(u,\dot{u}). \tag{2.3}$$

If the vector $\dot{u}^{\sigma}$ is of unit length i.e., $F(u,\dot{u}) = 1$ then the expression for the normal curvature k_n may be written as

$$k_n(u,\dot{u}) = \Omega^{*}_{\sigma\lambda}\dot{u}^{\lambda}\dot{u}^{\sigma}. \tag{2.4}$$

where the vector $\dot{u}^{\sigma}$ is of unit length. In view of (2.4), the quantities $\Lambda^{\varepsilon}_{\alpha\beta}$ satisfy the following condition

(2.5) a) $$\Lambda^{\varepsilon}_{\alpha\beta}\dot{u}^{\alpha} = k_n M^{\varepsilon}_{\beta}$$

b) $$g_{\varepsilon\gamma}\dot{u}^{\gamma}\Lambda^{\varepsilon}_{\alpha\beta} = -k_n M_{\alpha\beta},$$

where c) $$M^{\varepsilon}_{\beta} = g^{\varepsilon\gamma}M_{\gamma\beta}.$$

The mixed covariant derivative of an arbitrary tensor field T^i_α is given by

(2.6) $$\mathcal{B}_\gamma T^i_\alpha = \partial_\gamma T^i_\alpha - \dot{\partial}_\varepsilon T^i_\alpha \partial_\gamma G^\varepsilon - T^i_\varepsilon G^\varepsilon_{\alpha\gamma} + T^r_\alpha G^i_{rh} B^h_\gamma,$$

where $G^\varepsilon_{\alpha\gamma}$ are the induced Berwald connection parameters. In particular

(2.7) $$\mathcal{B}_\beta B^i_\alpha = V^i_{\alpha\beta} = B^i_{\alpha\beta} - B^i_\varepsilon G^\varepsilon_{\alpha\beta} + G^i_{hk} B^{hk}_{\alpha\beta}.$$

This equation can be rewritten as [126]

(2.8) $$V^i_{\alpha\beta} = N^{*i}\Omega^*_{\alpha\beta} - B^i_\varepsilon (\Lambda^\varepsilon_{\alpha\beta} + C^\varepsilon_{\alpha\beta|\sigma} \dot{u}^\sigma) + C^i_{hk|\gamma} y^\gamma B^{hk}_{\alpha\beta},$$

where $\Omega^*_{\alpha\beta}$ are the components of the second fundamental tensor. Gauss equations for Berwald curvature tensor, obtained by Sinha and Singh [126], are given by

(2.9) $$\begin{aligned} H_{\varepsilon\delta\beta\gamma} = {} & H_{hlkj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + (\Omega^*_{\varepsilon\beta}\Omega^*_{\delta\gamma} - \Omega^*_{\varepsilon\gamma}\Omega^*_{\delta\beta}) \\ & + 2M_{lh} B^h_\delta (\Omega^*_{\varepsilon\beta} V^l_{\gamma\sigma} - \Omega^*_{\varepsilon\gamma} V^l_{\beta\sigma}) \dot{u}^\sigma - (2N^{*h} C_{hlk|r} y^r \Omega^*_{\varepsilon\beta} B^{kl}_{\gamma\delta} \\ & - 2B^l_\delta g_{il} \Lambda^\sigma_{\varepsilon\beta} V^i_{\gamma\sigma} - 2\mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} g_{\alpha\delta} - 2\mathcal{B}_j C_{hlk|r} y^r B^{hlkj}_{\varepsilon\delta\beta\gamma} \\ & - 2\dot{\partial}_j C_{hlk|r} y^r B^{hlk}_{\varepsilon\delta\beta} V^j_{\gamma\sigma} \dot{u}^\sigma - 2C_{hlk|r} B^{hlk}_{\varepsilon\delta\beta} V^r_{\gamma\sigma} \dot{u}^\sigma \\ & - 2C_{hlk|r} y^r B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \qquad - \beta/\gamma). \end{aligned}$$

Associate Berwald curvature tensors of the Finsler space F_n and the hypersurface F_{n-1} being denoted by H_{hlkj} and $H_{\varepsilon\delta\beta\gamma}$ respectively.

3. Hypersurface of a Recurrent Finsler Space

A recurrent Finsler space F_n equipped with Berwald connection G^i_{jk} is

characterized by the condition

(3.1) $$\mathcal{B}_m H^i_{hkj} = \lambda_m H^i_{hkj}, \qquad H^i_{hkj} \neq 0 .$$

The non-zero covariant vector λ_m is the recurrence vector. This vector is independent of directional arguments y^i [89].

Let the hypersurface F_{n-1} of the recurrent Finsler space F_n be umbilical. Then the Gauss equations for umbilical hypersurface F_{n-1}, obtained by Singh and Chaubey [124], are as follows

(3.2) $$H_{\varepsilon\delta\beta\gamma} = H_{hlkj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + M^{*i} M^*_i (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta})$$

$$+ 2M_{\delta} k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma},$$

where

(3.3) a) $$M^{*i} M^*_i = k_n^2,$$

b) $$\Omega^*_{\alpha\beta}(u,\dot{u}) = k_n g_{\alpha\beta}(u,\dot{u})$$

and

c) $$P_{\varepsilon\delta\beta\gamma} = 2M_{lh} B^{hl}_{\delta\lambda} (\Lambda^{\lambda}_{\beta\sigma} \Omega^*_{\varepsilon\gamma} - \Lambda^{\lambda}_{\gamma\sigma} \Omega^*_{\varepsilon\beta}) \dot{u}^{\sigma}$$

$$- \{ 2N^{*h} C_{hlk|r} y^r \Omega^*_{\varepsilon\beta} B^{kl}_{\gamma\delta} - 2B^l_{\delta} g_{il} \Lambda^{\sigma}_{\varepsilon\beta} V^i_{\gamma\sigma} - 2\mathcal{B}_{\gamma} \Lambda^{\alpha}_{\varepsilon\beta} g_{\alpha\delta}$$

$$+ 2\mathcal{B}_j C_{hlk|r} y^r B^{hlkj}_{\varepsilon\delta\beta\gamma} + 2\dot{\partial}_j C_{hlk|r} y^r B^{hlk}_{\varepsilon\delta\beta} V^i_{\gamma\sigma} \dot{u}^{\sigma}$$

$$+ 2C_{hlk|r} B^{hlk}_{\varepsilon\delta\beta} V^r_{\gamma\sigma} \dot{u}^{\sigma} + 2C_{hlk|r} y^r B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \qquad - \beta/\gamma \}.$$

characterized by the condition

(3.1) $$\mathcal{B}_m H^i_{hkj} = \lambda_m H^i_{hkj}, \qquad H^i_{hkj} \neq 0.$$

The non-zero covariant vector λ_m is the recurrence vector. This vector is independent of directional arguments y^i [89].

Let the hypersurface F_{n-1} of the recurrent Finsler space F_n be umbilical. Then the Gauss equations for umbilical hypersurface F_{n-1}, obtained by Singh and Chaubey [124], are as follows

(3.2) $$H_{\varepsilon\delta\beta\gamma} = H_{hlkj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + M^{*i} M^*_i (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta})$$

$$+ 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma},$$

where

(3.3) a) $$M^{*i} M^*_i = k_n^2,$$

b) $$\Omega^*_{\alpha\beta}(u, \dot{u}) = k_n g_{\alpha\beta}(u, \dot{u})$$

and

c) $$P_{\varepsilon\delta\beta\gamma} = 2M_{lh} B^{hl}_{\delta\lambda} (\Lambda^\lambda_{\beta\sigma} \Omega^*_{\varepsilon\gamma} - \Lambda^\lambda_{\gamma\sigma} \Omega^*_{\varepsilon\beta}) \dot{u}^\sigma$$

$$- \{2N^{*h} C_{hlk|r} y^r \Omega^*_{\varepsilon\beta} B^{kl}_{\gamma\delta} - 2B^l_\delta g_{il} \Lambda^\sigma_{\varepsilon\beta} V^i_{\gamma\sigma} - 2\mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} g_{\alpha\delta}$$

$$+ 2\mathcal{B}_j C_{hlk|r} y^r B^{hlkj}_{\varepsilon\delta\beta\gamma} + 2\dot{\partial}_j C_{hlk|r} y^r B^{hlk}_{\varepsilon\delta\beta} V^i_{\gamma\sigma} \dot{u}^\sigma$$

$$+ 2C_{hlk|r} B^{hlk}_{\varepsilon\delta\beta} V^r_{\gamma\sigma} \dot{u}^\sigma + 2C_{hlk|r} y^r B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \qquad - \beta/\gamma\}.$$

The covariant derivative of $H_{\varepsilon\delta\beta\gamma}$ is given by

(3.4) $$\mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = (\mathcal{B}_m H_{hlkj} B^m_\theta + \frac{\partial H_{hlkj}}{\partial y^r} V^r_{\sigma\theta}\dot{u}^\sigma) B^{hlkj}_{\varepsilon\delta\beta\gamma}$$

$$+\mathcal{B}_\theta [M^{*i} M^*_i (g_{\beta\varepsilon} g_{\delta\gamma} - g_{\gamma\varepsilon} g_{\delta\beta}) + 2M_\delta k^2_n (g_{\varepsilon\beta} g_{\sigma\gamma}$$

$$- g_{\varepsilon\gamma} g_{\sigma\beta})\dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}] + H_{hlkj}(B^{lkj}_{\delta\beta\gamma} V^h_{\varepsilon\theta} + B^{hkj}_{\varepsilon\beta\gamma} V^l_{\delta\theta}$$

$$+ B^{hlj}_{\varepsilon\delta\gamma} V^k_{\beta\theta} + B^{hlk}_{\varepsilon\delta\beta} V^j_{\gamma\theta}).$$

Transvecting (3.1) by g_{il}, we get

(3.5) $$\mathcal{B}_m H_{hlkj} = \lambda_m H_{hlkj} + (\mathcal{B}_m g_{il}) H^i_{hkj}.$$

In view of the equation (3.5), we may write the equation (3.4) as

(3.6) $$\mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = [\lambda_m H_{hlkj} B^m_\theta + (\mathcal{B}_m g_{il}) H^i_{hkj} B^m_\theta + \frac{\partial H_{hlkj}}{\partial y^r}$$

$$V^r_{\sigma\theta}\dot{u}^\sigma] B^{hlkj}_{\varepsilon\delta\beta\gamma} + \mathcal{B}_\theta [M^{*i} M^*_i (g_{\beta\varepsilon} g_{\delta\gamma} - g_{\gamma\varepsilon} g_{\delta\beta})$$

$$+ 2M_\delta k^2_n (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta})\dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}] + H_{hlkj}(B^{lkj}_{\delta\beta\gamma}$$

$$V^h_{\varepsilon\theta} + B^{hkj}_{\varepsilon\beta\gamma} V^l_{\delta\theta} + B^{hlj}_{\varepsilon\delta\gamma} V^k_{\beta\theta} + B^{hlk}_{\varepsilon\delta\beta} V^j_{\gamma\theta}].$$

Using the equation (3.2) in the equation (3.6), we get

(3.7) $$\mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = \lambda_\theta [H_{\varepsilon\delta\beta\gamma} - M^{*i} M^*_i (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta})$$
$$- 2M_\delta k^2_n (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta})\dot{u}^\sigma - P_{\varepsilon\delta\beta\gamma}] + [(\mathcal{B}_m g_{il}) H^i_{hkj} B^m_\theta$$

$$+\frac{\partial H_{hlkj}}{\partial y^{r}} V_{\sigma\theta}^{r} \dot{u}^{\sigma}] B_{\varepsilon\delta\beta\gamma}^{hlkj} + \mathcal{B}_{\theta} [M^{*i} M_{i}^{*} (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\gamma\varepsilon} g_{\delta\beta})$$

$$+ 2M_{\delta} k_{n}^{2} (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma}] + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^{h}$$

$$+ B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^{l} + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^{k} + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^{j}),$$

where $\lambda_{\theta} = \lambda_{m} B_{\theta}^{m}$.

Let us assume

$$T_{\varepsilon\delta\beta\gamma} = H_{\varepsilon\delta\beta\gamma} - M^{*i} M_{i}^{*} (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) - 2M_{\delta} k_{n}^{2} \tag{3.8}$$

$$(g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} - P_{\varepsilon\delta\beta\gamma}.$$

In view of the equation (3.8), the equation (3.7) may be written as

$$\mathcal{B}_{\theta} T_{\varepsilon\delta\beta\gamma} = \lambda_{\theta} T_{\varepsilon\delta\beta\gamma} + [(\mathcal{B}_{m} g_{il}) H_{hkj}^{i} B_{\theta}^{m} + \frac{\partial H_{hlkj}}{\partial y^{r}} V_{\sigma\theta}^{r} \dot{u}^{\sigma}] \tag{3.9}$$

$$B_{\varepsilon\delta\beta\gamma}^{hlkj} + H_{hlkj} [B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^{h} + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^{l} + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^{k} + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^{j}].$$

Therefore

$$\mathcal{B}_{\theta} T_{\varepsilon\delta\beta\gamma} = \lambda_{\theta} T_{\varepsilon\delta\beta\gamma}$$

if and only if

$$[(\mathcal{B}_{m} g_{il}) H_{hkj}^{i} B_{\theta}^{m} + \frac{\partial H_{hlkj}}{\partial y^{r}} V_{\sigma\theta}^{r} \dot{u}^{\sigma}] B_{\varepsilon\delta\beta\gamma}^{hlkj} + H_{hlkj} [B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^{h} \tag{3.10}$$

$$+ B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^{l} + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^{k} + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^{j}] = 0.$$

Hence, we conclude

Theorem 3.1. *If an umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space F_n whose recurrence vector field λ_m is not normal to the hypersurface F_{n-1}, then $T_{\varepsilon\delta\beta\gamma}$ is recurrent with the recurrence vector $\lambda_\theta = \lambda_m B^m_\theta$ if and only if* (3.10) *holds good.*

Suppose

$$(3.11)\qquad J_{\varepsilon\delta\beta\gamma} = (M^{*\iota} M^*_\iota)(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) + 2M_\delta k^2_n (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta})\dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}.$$

In view of the equation (3.8), the equation (3.11) may be written as

$$(3.12)\qquad H_{\varepsilon\delta\beta\gamma} = T_{\varepsilon\delta\beta\gamma} + J_{\varepsilon\delta\beta\gamma}.$$

This equation shows that $H_{\varepsilon\delta\beta\gamma}$ is recurrent with the recurrence vector field λ_θ if the tensors $T_{\varepsilon\delta\beta\gamma}$ and $J_{\varepsilon\delta\beta\gamma}$ are recurrent with the same recurrence vector field λ_θ. This fact and the theorem 3.1 prove the following

Theorem 3.2. *The sufficient conditions that the curvature tensor $H_{\varepsilon\delta\beta\gamma}$ of an umbilical hypersurface immersed in a recurrent space, whose recurrence vector field λ_m is not normal to the hypersurface F_{n-1}, be recurrent with the recurrence vector field λ_θ are that the relation* (3.10) *holds and $J_{\varepsilon\delta\beta\gamma}$ is recurrent with the recurrence vector field λ_θ.*

Now, we assume that the recurrence vector field λ_m is normal to the hypersurface F_{n-1} i.e., $\lambda_m B^m_\theta = \lambda_\theta$. The Bianchi identity for a recurrent Finsler space is given by [89]

$$(3.13)\qquad \lambda_r H^i_{hkj} + \lambda_j H^i_{hrk} + \lambda_k H^i_{hjr} = 0.$$

Transvecting (3.13) by g_{il}, we get

(3.14) $$\lambda_r H_{hlkj} + \lambda_j H_{hlrk} + \lambda_k H_{hljr} = 0.$$

Multiplying the above equation by $B^{hlkj}_{\varepsilon\delta\beta\gamma}$ and using the conditions $\lambda_\theta = \lambda_m B^m_\theta = 0$ and $\lambda_r \neq 0$, we get

(3.15) $$H_{hlkj} B^{hlkj}_{\varepsilon\delta\beta\gamma} = 0.$$

In view of the equation (3.15), the equation (3.2) may be written as

(3.16) $$H_{\varepsilon\delta\beta\gamma} = M^{*i} M^*_i (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) + 2M_\delta k^2_n (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}$$

This leads to

Theorem 3.3. *If an umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space F_n whose recurrence vector λ_m is normal to the hypersurface F_{n-1}, then the tensor $T_{\varepsilon\delta\beta\gamma}$ defined by (3.8) vanishes.*

Transvecting equation (3.16) by $\dot{u}^\delta$ and using the equation (4.2.12b), we get

(3.17) $$H_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = M^{*i} M^*_i (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \dot{u}^\delta + P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta.$$

Transvecting the equation (3.3c) by $\dot{u}^\delta$ and using the equations $M_{i|h} y^h = 0$, (1.5.2a) and (1.6.11a), we get

(3.18) $$P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = -2B^l_\delta g_{il} \Lambda^\sigma_{\varepsilon\beta} V^i_{\gamma\sigma} \dot{u}^\delta - 2\mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} g_{\alpha\delta} \dot{u}^\delta \quad - \beta / \gamma.$$

In view of the equation (2.8), equation (3.18) may be written as

(3.19) $$P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = [-2B^l_\delta g_{il} \dot{u}^\delta \Lambda^\sigma_{\varepsilon\beta} \{N^{*i} \Omega^*_{\gamma\sigma} - B^i_\lambda (\Lambda^\lambda_{\gamma\sigma} + C^\lambda_{\gamma\sigma/\eta} \dot{u}^\eta)$$

$$+ C^i_{hk|\eta} y^\eta B^{hk}_{\gamma\sigma}\} - 2\mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} g_{\alpha\delta} \dot{u}^\delta \quad - \beta/\gamma].$$

Using equations (4.2.4a), (1.3.4), (1.5.2b) and (1.6.11a) in the equation (3.19), we have

$$P_{\varepsilon\delta\beta\gamma}\dot{u}^{\delta} = [2B^{l}_{\delta}g_{il}\dot{u}^{\delta}\Lambda^{\sigma}_{\varepsilon\beta}B^{i}_{\lambda}(\Lambda^{\lambda}_{\varepsilon\sigma} + C^{\lambda}_{\gamma\sigma|\eta}\dot{u}^{\eta} - 2\mathcal{B}_{\gamma}\Lambda^{\alpha}_{\varepsilon\beta}g_{\alpha\delta}\dot{u}^{\delta} \quad - \beta/\gamma]$$

In view of (4.2.2), above equation implies

$$P_{\varepsilon\delta\beta\gamma}\dot{u}^{\delta} = 2g_{\alpha\delta}\dot{u}^{\delta}\Lambda^{\sigma}_{\varepsilon\beta}[\Lambda^{\alpha}_{\gamma\sigma} + C^{\alpha}_{\gamma\sigma|\eta}\dot{u}^{\eta}] - 2\mathcal{B}_{\gamma}\Lambda^{\alpha}_{\varepsilon\beta}g_{\alpha\delta}\dot{u}^{\delta} \quad - \beta/\gamma. \tag{3.20}$$

Using the equation (3.20) in the equation (3.17), we get

$$H_{\varepsilon\delta\beta\gamma}\dot{u}^{\delta} = M^{*i}M^{*}_{i}(g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta})\dot{u}^{\delta} \tag{3.21}$$

$$+ 2g_{\alpha\delta}\dot{u}^{\delta}[\Lambda^{\sigma}_{\varepsilon\beta}(\Lambda^{\alpha}_{\gamma\sigma} + C^{\alpha}_{\gamma\sigma|\eta}\dot{u}^{\eta}) - \mathcal{B}_{\gamma}\Lambda^{\alpha}_{\varepsilon\beta} \quad - \beta/\gamma]$$

The hypersurface F_{n-1} of constant curvature is characterized by the equation

$$H_{\varepsilon\delta\beta\gamma}\dot{u}^{\delta} = K(g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta})\dot{u}^{\delta}. \tag{3.22}$$

From equations (3.21) and (3.22), we conclude

Theorem 3.4. *If an umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space F_n whose recurrence vector field λ_m is normal to F_{n-1}, then the necessary and sufficient condition that F_{n-1} be a space of constant curvature is that*

$$g_{\alpha\delta}\dot{u}^{\delta}[\mathcal{B}_{\gamma}\Lambda^{\alpha}_{\varepsilon\beta} - \Lambda^{\sigma}_{\varepsilon\beta}(\Lambda^{\alpha}_{\gamma\sigma} + C^{\alpha}_{\gamma\sigma|\eta}\dot{u}^{\eta}) \quad - \beta/\gamma] = 0.$$

If the hypersurface F_{n-1} is of minimal variety then mean curvature vector will vanish and equations (3.3a) and (3.3b) give

$$k_n = 0, \qquad \Omega^{*}_{\alpha\beta} = 0. \tag{3.23}$$

In view of the equation (3.23), the equation (2.2) implies

(3.24) $$\Lambda^{\varepsilon}_{\beta\gamma} = 0 .$$

Using the equation (3.24) in the equation (3.3c), we get

(3.25) $$P_{\varepsilon\delta\beta\gamma} = 2\,\mathcal{B}_j\, C_{hlk|r}\, y^r B^{hlkj}_{\varepsilon\delta\beta\gamma} + 2\dot{\partial}_j C_{hlk|r}\, y^r B^{hlk}_{\varepsilon\delta\beta}$$

$$V^i_{\gamma\sigma}\dot{u}^{\sigma} + 2C_{hlk|r} B^{hlk}_{\varepsilon\delta\beta} V^r_{\gamma\sigma}\dot{u}^{\sigma} + 2C_{hlk|r}\, y^r B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \quad - \beta/\gamma .$$

Using equations (1.7.9), (2.8), (3.23) and (3.24) in the equation (3.25), we find

(3.26) $$P_{\varepsilon\delta\beta\gamma} = 2(\mathcal{B}_j\mathcal{B}_k\, g_{hl}) B^{hlkj}_{\varepsilon\delta\beta\gamma} + 2\dot{\partial}_j C_{hlk|r}\, y^r B^{hlk}_{\varepsilon\delta\beta}$$

$$(C^l_{ms|\eta}\, y^{\eta} B^{ms}_{\gamma\sigma}\dot{u}^{\sigma} - B^l_{\eta} C^{\eta}_{\gamma\sigma|\lambda}\dot{u}^{\lambda}\dot{u}^{\sigma}) + 2C_{hlk|r} B^{hlk}_{\varepsilon\delta\beta}$$

$$(C^r_{ms|\eta}\, y^{\eta} B^{ms}_{\gamma\sigma}\dot{u}^{\sigma} - B^r_{\eta} C^{\eta}_{\gamma\sigma|\lambda}\dot{u}^{\lambda}\dot{u}^{\sigma}) + 2(\mathcal{B}_k\, g_{hl}) B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \quad - \beta/\gamma .$$

Using equations (1.9.1), (1.5.2a), (1.6.11a), (3.15) and $\dot{u}^{\sigma}_{|\lambda} = 0$ in the equation (3.26), we get

$$P_{\varepsilon\delta\beta\gamma} = C_{rhl} H^r_{kj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + \{2(\mathcal{B}_k\, g_{hl}) B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \quad - \beta/\gamma\},$$

which may be written as

(3.27) $$P_{\varepsilon\delta\beta\gamma} = C_{rhl} H^r_{kj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + \{2\,\mathcal{B}_k(\dot{\partial}_h y_l) B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \quad - \beta/\gamma\}.$$

Using the equation (1.7.10) in the equation (3.27), we get

(3.28) $$P_{\varepsilon\delta\beta\gamma} = C_{rhl} H^r_{kj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + (y_r G^r_{hkl} B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \quad - \beta/\gamma) .$$

In view of the equations (3.28) and (3.23), the equation (3.16) gives

$$H_{\varepsilon\delta\beta\gamma} = C_{rhl} H^r_{kj} B^{hlkj}_{\varepsilon\delta\beta\gamma} + (y_r G^r_{hkl} B^{lk}_{\delta\beta} V^h_{\gamma\varepsilon} \quad - \beta/\gamma) .$$

Thus, we conclude

Theorem 3.5. *If an umbilical hypersurface F_{n-1} is of minimal variety and is immersed in a recurrent space F_n whose recurrence vector field is normal to F_{n-1}, then the space F_{n-1} is Minkowskian (i.e., $H_{\varepsilon\delta\beta\gamma}=0$) if and only if*

$$C_{rhl}H^{r}_{kj}B^{hlkj}_{\varepsilon\delta\beta\gamma}=-y_r G^{r}_{hkl}B^{lk}_{\delta\beta}V^{h}_{\gamma\varepsilon} \quad -\beta/\gamma\,.$$

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REFERENCES

1. **Adati, T.:** On subprojective spaces 1, *Tohoku Math.*, **1, 2** (1957) 159-173.

2. **Adati, T.** and **Miyazawa, T.:** On a Riemannian space with recurrent conformal curvature, *Tensor N.S.*, **18** (1967) 348-354.

3. **Adati, T.** and **Miyazawa, T.:** On a projective transformations projective recurrent spaces, *Tensor N.S.*, **31** (1977) 49-54.

4. **Aikou, Tadashi:** Some remarks on conformal changes of generalized Finsler metrics, *Rep. Fac. Sci. Kagashima Univ. Math. Phy. Chem.*, **20** (1987) 57-61.

5. **Akbar-Zadeh, M.H. :** Les espaces de Finsler et certaines de leurs généralisations, *Ann. Scient Éc. Norm. Sup.*, **(3) 80** (1963) 1-79.

6. **Amur, K.** and **Desai, P. :** Recurrent properties of projective related spaces, *Tensor N.S.*, **29** (1975) 40-42.

7. **Amur, K.** and **Desai, P.:** Recurrent properties of spaces whose connections preserve parallelism, *Tensor N.S.*, **30** (1976) 92-94.

8. **Amur, K.** and **Desai, P.:** On symmetric spaces, *Tensor N.S.*, **29** (1975) 119-124.

9. **Berwald, L.:** Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden parallelismus, *Math Z.*. **25** (1926) 40-73; *Math. Z.*, **26** (1927) 176.

10. **Berwald, L.:** On Finsler and Cartan geometries III Two dimensional Finsler spaces with rectilinear extremals, *Annals of Math*, **42** (1941) 84-112.

11. **Berwald, L.:** Über die Beziehungen zwischen den theorien der parallelübertrangung in Finslerschen Räumen, *Nederl, Akad. Wetensch. Proc. Sér. A.*, **49** (1946) 642-647.

12. **Berwald, L.:** Über Finslersche und Cartansche Geometrie IV. projektive - Krümmung allgemeiner affiner Räume und Finslersche Räume Skalarer. Krümmung, *Ann. Math.*, **(2) 48** (1947) 755-781.

13. **Brown, C. M.:** A study of tensors which characterize a hypersurface of Finsler space, *Canad. J. Math.*, **20** (1968) 1025-1036.

14. **Cartan, Élie:** Sur les espaces de Finsler, *C. R. Acad. Sci. (Paris)*, **196** (1933) 582-586.

15. **Cartan, Élie:** Les espaces de Finsler, *Actualités Paris*, **79** (1934) 2nd edit 1971.

16. **Chaki, M.C.** and **Roy Chaudhery, A.N.:** On Ricci recurrent spaces of second order, *J. Ind. Math. Soc.*, **9** (1967) 279-287.

17. **Comic, Irena:** Recurrent curvature tensors in Finsler spaces with generalized connection, *Publ. Math. Debrecen,* **35** (**1-2**) (1988) 83-91.

18. **Comic, Irena:** Geodesic curves and paths in the recurrent Finsler spaces, F_n, *The proc. of Math.*, (1988) 119-131.

19. **Davies, E.T.:** Lie derivation in generalised metric spaces, *Ann. Mat. Pura Appl.*, (**4**), **18** (1939) 261-274.

20. **Davies, E.T.:** Subspaces of a Finsler space, *Proc. Lond. Math. Soc.*, **44** (1945) 19-39.

21. **Dikshit, Shalini:** *Certain types of Recurrences in Finsler spaces,* D. Phil. Thesis, University of Allahabad, Allahabad, 1992.

22. **Dubey, R.S.D.** and **Singh, Hukum:** Hypersurfaces of Recurrent Finsler spaces, *Indian J. Pure Appl. Math.*, **11** (**19**) (1980) 1125-1129.

23. **Dubey, R.S.D.** and **Srivastava, A.K.:** On recurrent Finsler spaces, *Bull. Soc. Math. Belgique*, **33** (1981) 283-288.

24. **Dwivedi, V.J.:** *Certain types of transformations in Finsler spaces*, D. Phil. Thesis, University of Allahabad, Allahabad, 1984.

25. **Hashiguchi, M.:** On conformal transformations of Finsler metrices, *J. Math. Kyoto Univ.*, **19** (1976) 25-50.

26. **Heil, E.:** Eine charakterisierung lokat Minkowskischer Räume, *Math. Annalen*, **167** (1966) 64-70.

27. **Izumi, H.:** On *P-Finsler spaces I, *Defense Academy of Japan*, **16** (**4**) (1976) 133-138.

28. **Izumi, H.:** On *P-Finsler spaces II, *Defense Academy of Japan*, **17** (**1**) (1977) 1-9.

29. **Izumi, H.:** Conformal transformations of Finsler spaces I, Concircular transformations of a curve with Finsler metric, *Tensor N.S.*, **31** (1977) 23-41.

30. **Izumi, H.:** Conformal transformations of Finsler spaces II, An *h*-conformally flat Finsler space, *Tensor N.S.*, **34** (1980) 337-359.

31. **Izumi, H.:** On *P-Finsler space of scalar curvature, *Tensor N.S.*, **38** (1982) 220-222.

32. **Izumi, H.** and **Toshida, Mamoru:** On Finsler spaces of perpendicular scalar curvature, *Tensor N.S.*, **32** (1978) 220-224.

33. **Izumi, H.** and **Srivastava, T.N.:** On R3-like Finsler spaces, *Tensor N.S.*, **32** (1978) 339-349.

34. **Kawaguchi, H.:** On Finsler space with the vanishing second curvature tensor, *Tensor N.S.*, **26** (1972) 250-254.

35. **Knebelman, M.S.:** Conformal geometry of generalized metric spaces, *Proc. Nat. Acad. Sci. (U.S.A.)*, **15** (1929) 376-379.

36. **Matsumoto, M.:** Affine transformations of Finsler spaces, *J. Math. Kyoto Univ.*, **3** (1963) 1-35.

37. **Matsumoto, M.:** A geometric meanings of a concept of isotropic Finsler spaces, *J. Math. Kyoto Univ.*, **9** (1969) 405-411.

38. **Matsumoto, M.:** The theory of Finsler connections, *Publ. of the study Group of Geometry 5, College of Liberal Arts and Sci., Okayama Univ., Japan* (1970).

39. **Matsumoto, M.:** On h-isotropic and C^h-recurrent Finsler spaces, *J. Math. Kyoto Univ.*, **11** (1971) 1-9.

40. **Matsumoto, M.:** On some transformations of locally Minkowskian spaces, *Tensor N.S.*, **22** (1971) 103-111.

41. **Matsumoto, M.:** On Finsler spaces with curvature tensors of some special forms. *Tensor N.S.*, **22** (1971) 201-204.

42. **Matsumoto, M.:** On C-reducible Finsler spaces, *Tensor N.S.*, **24** (1972) 29-37.

43. **Matsumoto, M.:** A theory of three-dimensional Finsler spaces in terms of scalars, *Demontratio Mathematica*, **VI** (1973) 223-251.

44. **Matsumoto, M.:** On Finsler spaces with Rander's metric and special form of important tensors, *Journal of Mathematics of Kyoto University*, **14** **(3)** (1974) 477-492.

45. **Matsumoto, M.:** Finsler space admitting a concurrent vector field, *Tensor N.S.*, **28** (1974) 239-249.

46. **Matsumoto, M.:** On three dimensional Finsler spaces satisfying the T-and β^{P}-conditions, *Tensor N.S.*, **29** (1975) 13-20.

47. **Matsumoto, M.:** On the indicatrices of a Finsler space, *Periodica Math. Hungar,* **88** (**2**) (1977) 187-193.

48. **Matsumoto, M.:** Fundamental functions of S3-like Finsler spaces, *Tensor N.S.*, **34** (1980) 141-146.

49. **Matsumoto, M.:** Projective changes of Finsler metrics and projectively flat Finsler spaces, *Tensor N.S.*, **34** (1980) 303-315.

50. **Matsumoto, M.:** The induced and intrinsic Finsler connection and Finslerian projective geometry, *J. Math. Kyoto. Univ.*, **25** (1985) 107-144.

51. **Matsumoto, M.** and **Numata, S.:** On semi-C-reducible Finsler spaces with constant coefficients and C2-like Finsler spaces, *Tensor N.S.*, **34** (1980) 218-222.

52. **Matsumoto, M.** and **Shibata, Chākō:** On semi-C-reducibility T-tensor = 0 and S4-likeness of Finsler spaces, *J. Math. Kyoto Univ.*, **19** (1979) 301-304.

53. **Meher, F.M.:** *Certain investigations in Finsler spaces,* Ph. D. Thesis, Berhampur Univ. India, 1972.

54. **Meher, F.M.:** Projective motion in a symmetric Finsler space, *Tensor N.S.*, **23** (1972) 275-278.

55. **Misra, R.B.:** Projective tensors in a conformal Finsler space, *Acad. Roy. Belg. Bull. Cl. Sci.*, (**5**) **52** (1966) 1275-1279.

56. **Misra, R.B.:** A symmetric Finsler space, *Tensor N.S.*, **24** (1972) 346-350.

57. **Misra, R.B.:** A projective symmetric Finsler space, *Math. Z.*, **126** (1972) 143-153.

58. **Misra, R.B.:** On a recurrent Finsler space, *Rev. Roumaine Math. Pures Appl.*, **18** (1973) 701-712.

59. **Misra, R.B.:** A birecurrent Finsler manifold with affine motion, *Indian J. Pure. Appl. Math.*, **6** (1975) 1441-1448.

60. **Misra, R.B.:** A turning point in the theory of recurrent Finsler manifolds, *J. South Gujarat Univ.*, **6** (1977) 72-96.

61. **Misra, R.B.:** A turning point in the theory of recurrent Finsler manifolds II. Certain types of projective motions, *Bull. Un. Math. Ital.*, (5) (1979) 16-8.

62. **Misra, R.B., Kishore, N.** and **Pandey, P.N.:** Projective motion in SNP-F_n, *Bull. U.M.I.*, **14** (A) (5) (1977) 513-519.

63. **Misra, R.B.** and **Meher, F.M.:** A SHR-F_n admitting an affine motion, *Acta. Math. Acad. Sci. Hungar.*, **22** (1971) 423-429.

64. **Misra, R.B.** and **Meher, F.M.:** A Finsler space with a special concircular projective motion, *Tensor N.S.*, **24** (1972) 288-292.

65. **Mishra, R.S.** and **Pande, H.D.:** Recurrent Finsler spaces, *J. Indian Math. Soc., N.S.*, **32** (1968) 17-22.

66. **Miyazawa, T.:** On conformal transformations of conformally recurrent spaces, *Tru Mathematics*, **12** (1976) 25-28.

67. **Miyazawa, T.:** On projectively recurrent spaces, *Tensor N.S.*, **32** (1978) 216-218.

68. **Miyazawa, T.** and **Chuman, Gorō:** On certain subspaces of Riemannian recurrent spaces, *Tensor N.S.*, **23** (1972) 253-260.

69. **Mogi, I.:** A remark on recurrent curvature spaces, *Kodai Math. Sem. Rep.*, (1950) 73-79.

70. **Moór, A.:** Untersuchungen Über Finslerräume von rekurrenter Krümmung, *Tensor N.S.*, **13** (1963) 1-18.

71. **Moór, A.:** Räume von zweifach rekurrenter krümmung, *Publ. Math. Debrecen*, **18** (1971) 165-169.

72. **Moór, A.:** Über Finslerräume von zweifach rekurrenter Krümmung, *Acta. Math. Acad. Sci. Hungar*, **22** (1971) 453-465.

73. **Moór, A.:** Unterräume von rekurrenter Krümmung in Finslerrmen, *Tensor N.S.*, **24** (1972) 261-265.

74. **Moór, A.:** Finslerräume von rekurrenter Torsion, *Publ. Math. Debrecen*,**21** (1974) 255-265.

75. **Moór, A.:** Über eine Übertragungs theorie der metrischer Linienel ementräumemit rekurrent Grund tensor, *Tensor N.S.*, **29** (1975) 47-63.

76. **Pande, H.D.:** The conformal transformation in a Finsler space, *Math. Nachrichten*, **41** (1969) 247-252.

77. **Pande, H.D.** and **Khan, T.A.:** Recurrent Finsler spaces with Cartan's first curvature tensor field, *Atti. Accad. Naz. Lincei, Rendiconti*, **55** (1973) 224-227.

78. **Pande, H.D.** and **Khan, T.A.:** General decomposition of Berwald's curvature tensor field in recurrent Finsler space, *Atti. Accad. Naz. Lincei Rendi, Cl. Sci. Mat. nature.*, **55** (1973) 680-685.

79. **Pande, H.D.** and **Tiwari, S.K.:** Recurrent Finsler spaces, *J. Nat. Acad. Math.*, **Vol. II** (1997) 98-109.

80. **Pandey, P.N.:** Conformal covariant derivative in a Finsler manifold, *Atti. Accad. Sci. Lett. Arti. Polermo Parte I.*, **37** (**4**) (1977-78) 341-350.

81. **Pandey, P.N.:** CA-collineation in a birecurrent Finsler manifold, *Tamkang, J. Math.*, **9, 1** (1978) 79-81.

82. **Pandey, P.N.:** Groups of conformal transformations in conformally related Finsler manifolds, *Atti. Accad. Nat. Lincei Rendiconti*, **65** (1978) 269-274.

83. **Pandey, P.N.:** Decomposition of curvature tensor in a recurrent Finsler manifold, *Tamkang J. Math.*, **10** (**1**) (1979) 31-34.

84. **Pandey, P.N.:** A recurrent Finsler manifold admitting special recurrent transformations, *Prog. Math.*, **13** (**1-2**) (1979) 85-98.

85. **Pandey, P.N.:** Affine motion in a recurrent Finsler manifold, *Ann. Fac. Sci. de Kinshasa Sec. Math. Phys.*, **6** (**1-2**) (1980) 51-63.

86. **Pandey, P.N.:** On NPR-Finsler manifolds, *Ann. Fac. Sci. de Kinshasa Sec. Math. Phys.*, **6** (**1-2**) (1980) 65-77.

87. **Pandey, P.N.:** A recurrent Finsler manifold with a concircular vector field, *Acta. math. Acad. Sci. Hung.*, **35** (**3-4**) (1980) 465-466.

88. **Pandey, P.N.:** On decomposability of curvature tensor of a Finsler manifold, *Acta. Math. Acad. Sci. Hung.*, **38** (**1-4**) (1981) 109-116.

89. **Pandey, P.N.:** A note on recurrence vector, *Proc. Nat. Acad. Sci.* (India) **51A, I,** (1981) 6-8.

90. **Pandey, P.N.:** Some identities in an NPR-Finsler manifold, *Proc. Nat. Acad. Sci.* (India) **51A** (1981) 185-189.

91. **Pandey, P.N.:** *Groups of transformations in Finsler manifolds*, D. Phil. Thesis, University of Allahabad, 1982.

92. **Pandey, P.N.:** On Lie-recurrent Finsler manifolds, *Indian J. mathematics*, **24** (1982) 135-143.

93. **Pandey, P.N.:** On a Finsler space of zero projective curvature, *Acta. Math. Acad. Sci. Hung.*, **39** (**4**) (1982) 387-388.

94. **Pandey, P.N.:** Certain types of projective motions in a Finsler manifold, *Atti. Accad. Poleritana die pericolanti Cl. Sci. Fis. Math. Nat.*, **60** (1982-83) 287-300.

95. **Pandey, P.N.:** On some Finsler spaces of scalar curvature, *Prog. Math.*, **18** (**1**) (1984) 41-48.

96. **Pandey, P.N.:** On Wong's conjecture, *Prog. Math.*, **18** (**2**) (1984) 89-94.

97. **Pandey, P.N.:** A symmetric Finsler manifold with a concircular vector field, *Proc. Nat. Acad. Sci.*, (India), **54A** (1984) 271-273.

98. **Pandey, P.N.:** Projective motion in symmetric and projectively symmetric Finsler manifolds, *Proc. Nat. Acad. Sci.* (India), **54A** III (1984) 274-278.

99. **Pandey, P.N.:** On birecurrent affine motion in a Finsler manifold, *Acta. Math. Acad. Sci. Hung.*, **45** (**3-4**) (1985) 251-260.

100. **Pandey, P.N.:** Certain types of affine motion in a Finsler Manifold I, *Colloquium Mathematicum*, **49** (**2**) (1985) 243-252.

101. **Pandey, P.N.:** A recurrent Finsler manifold with a torseforming vector field, *Atti. Accad. Sci. Tarino*, **119** (1985) 100-106.

102. **Pandey, P.N.:** Certain types of projective motion in a Finsler manifold II. *Atti. della Accadamia delle. Scianza di Torino*, **120** (**5-6**) (1986) 168-178.

103. **Pandey, P.N.:** Certain types of affine motion in Finsler manifold II, *Colloquium Mathematicum*, **53** (**2**) (1987) 219-227.

104. **Pandey, P.N.:** Certain types of affine motion in a Finsler manifold III, *Colloquium Mathematicum*, **56** (**2**) (1988) 333-340.

105. **Pandey, P.N.:** On decomposability of curvature tensor of a Finsler manifold II, *Proc. Nat. Acad. Sci. (India)*, **58A** (1988) 85-88.

106. **Pandey, P.N.:** *Some Problems in Finsler spaces*, D. Sc. Thesis, University of Allahabad, Allahabad, (India), 1993.

107. **Pandey, P.N.** and **Dikshit, Shalini:** On *P- and P-reducible Finsler spaces of recurrent curvature, *J. Int. Acad. Physical Sciences*, **2** (1998) 11-17.

108. **Pandey, P.N.** and **Dwivedi, V.J.:** Projective motion in an RNP-Finsler spaces, *Tamkang J. Math.*, **17** (**1**) (1986) 87-98.

109. **Pandey, P.N.** and **Dwivedi, V.J.:** On T-recurrent Finsler spaces, *Prog. Math.*, **21** (**2**) (1987) 101-112.

110. **Pandey, P.N.** and **Dwivedi, V.J.:** Affine motion in T-recurrent Finsler manifold, *Proc. Nat. Acad. Sci.* (India), **57A IV** (1987) 438-446.

111. **Pandey, P.N.** and **Dwivedi, V.J.:** On curvature Collineation in an NP-Finsler space, *Proc. of the 3rd conference of I.A.P.S.*, (1999) 201-213.

112. **Pandey, P.N.** and **Misra, R.B.:** Projective recurrent Finsler spaces, *Publications Mathematicae Debrecen*, **28** (**3-4**) (1981) 191-198.

113. **Pandey, P.N.** and **Verma, Reema:** On C^h-birecurrent Finsler spaces, *J. Int. Acad. Physical Sciences*, **2** (1998) 43-50.

114. **Pandey, P.N.** and **Verma, Reema:** Infinitesimal affine motions in a Finsler space equipped with Cartan's connection, *J. Int. Acad. Physical Sciences*, **1** (1997) 39-54.

115. **Pandey, P.N.** and **Verma, Reema:** On projective transformations between special Finsler spaces, *J. Int. Acad. Physical Sciences*, **1** (1997) 183-193.

116. **Patterson, E.M.:** Some theorems on Ricci recurrent spaces, *J. Lond. Math. Soc.*, **27** (1952) 285-295.

117. **Prasad, B.N., Gupta, B.N.** and **Singh, D.D.:** Conformal transformation in Finsler spaces with an (α, β) metric, *Indian J. Pure Appl. Math.*, **18** (**4**) (1987) 290-301.

118. **Prasad, B.N., Singh, V.P.** and **Singh, J.N.:** Conformal correspondence of Finslerian hypersurfaces, *Indian J. Pure Appl. Math.*, **18** (1987) 12-26.

119. **Rastogi, S.C.:** On Finsler space of recurrent curvature tensors, *Kyungpook Math.*, **20** (**1**) (1980) 37-45.

120. **Rund, H.:** *The differential geometry of Finsler spaces,* Springer Verleg, Berlin-Göttingen-Heidelberg, 1959.

121. **Rund, H.:** Intrinsic and induced curvature theories of subspace of Finsler space, *Tensor N.S.*, **16** (1965) 294-312.

122. **Ruse, H.S.:** Three dimensional spaces of recurrent curvature, *Proc. Lond. Math. Soc.*, **50** (1949) 438-446.

123. **Sen, R.N.:** Finsler spaces of recurrent curvature, *Tensor N.S.*, **19** (1968) 291-299.

124. **Singh, U.P.** and **Chaubey, G.C.:** On hypersurface of recurrent Finsler spaces, *Indian J. Pure Appl. Math.*, (1977) 497-504.

125. **Singh, U.P.** and **Singh, U.B.:** Umbilical subspaces of recurrent Finsler spaces, (1976).

126. **Sinha, B.B.** and **Singh, S.P.:** A generalization of Gauss and Codazzi equations for Berwald curvature tensor, *Tensor N.S.*, **22** (1971) 112-116.

127. **Sinha, B.B.** and **Singh, S.P.:** On recurrent spaces of second order in Finsler spaces, *Yakohama Math. J.*, **18** (1970) 27-32.

128. **Sinha, R.S.:** Affine motion in a recurrent Finsler space, *Tensor, N.S.*, **20** (1969) 261-264.

129. **Sinha, R.S.:** On projective motion in a Finsler space with recurrent curvature, *Tensor, N.S.*, **21** (1970) 124-126.

130. **Sinha, R.S.:** Infinitesimal projective transformation in a Finsler spaces, *Prog. Math.*, **5** (1971) 30-34.

131. **Sinha, R.S.** and **Faruqui, S.A.:** On Projective related recurrent and projective recurrent Finsler space, *Indian J. Pure Appl. Math.*, **10** (4) (1979) 394.

132. **Soós, Gy.:** Über einfach Finslersche Räume, *Publ. Math. Debrecen*, **7** (1960) 364-373.

133. **Takano, K.:** On the projective motion in a space with recurrent curvature, *Tensor N.S.*, **12** (1962) 28-32.

134. **Takano, K.:** On the existence of affine motion in a space with recurrent curvature, *Tensor N.S.*, **17** (1966) 68-73.

135. **Takano, K.:** On the existence of affine motion in a space with recurrent curvature II, *Tensor N.S.*, **17** (1966) 212-216.

136. **Takano, K.:** Recurrent affine motion in $S-AK_N^*$-space, *Tensor N.S.*, **17** (1966) 321-326.

137. **Takano, K.:** Decomposition of curvature tensor in a recurrent space, *Tensor N.S.*, **18** (1967) 343-347.

138. **Verma, Reema:** *Some transformations in Finsler spaces*, D. Phil. Thesis, University of Allahabad, Allahabad (India), 1991.

139. **Walker, A.G.:** On Ruse's space of recurrent curvature, *Proc. Lond. Math. Soc.*, **52** (1950) 36-64.

140. **Wong, Y.C.:** A class of non-Riemannian K^*-spaces, *Proc. Lond. Math. Soc.*, **3** (1953) 118-128.

141. **Wong, Y.C.:** Recurrent tensor on a linearly connected differentiable manifold, *Trans. Amer. Math. Soc.*, **99** (1961) 325-341.

142. **Wong, Y.C.:** Linear connections with zero torsion and recurrent curvature, *Trans. Amer. Math. Soc.*, **102** (1962) 471-506.

143. **Wong, Y.C.** and **Yano, K.:** Projectively flat spaces with recurrent curvature, *Comment. Math. Helv.*, **35** (1961) 223-232.

144. **Yablokov, D.M.:** Conformal transformations in a Finsler space, *Izv. Vyssh. Uchebm. Zaved Mat.*, (**7**) (1989) 53-59.

145. **Yamaguchi, S.:** On infinitesimal projective transformations in non-Riemannian recurrent spaces, *Tensor N.S.*, **18** (1967) 271-278.

146. **Yang, W.M.** and **Liu, Y.T.:** Riemann spaces with recurrent projective curvature tensor,*Wuhan dacue Xuebao*, (1979) 11-19.

147. **Yano, K.:** On the torse-forming directions in Riemannian spaces, *Proc. Imp. Acad. Takyo*, **20** (1944) 340-345.

148. **Yano, K.:** *The theory of Lie derivatives and its applications*, North-Holand Publishing Co., Amsterdam, 1957.

149. **Yaseen Abdo Qasem, Fahmi:** *On transformations in Finsler spaces*, D. Phil. Thesis, University of Allahabad, Allahabad (India), 2000.

150. **Yoshida, Mamoru:** On an R3-like Finsler space and its special cases, *Tensor N.S.*, **34** (1980) 157-166.

*****